

ΑΝΙΣΩΣΕΙΣ ΚΑΙ ΑΚΡΟΤΑΤΑΠΑΡΑΔΕΙΓΜΑΤΑ

1.

Να αποδείξετε ότι για κάθε $x \in \mathbb{R}$ ισχύει: $e^x \geq x+1$



Θεωρώ τη συνάρτηση $f(x) = e^x - x - 1$

$$f'(x) = (e^x - x - 1)' = (e^x)' - (x)' - (1)' = e^x - 1 - 0 = e^x - 1$$

$$f'(x) > 0 \iff e^x - 1 > 0 \iff e^x > 1 \iff e^x > e^0 \iff x > 0$$

$$f'(x) = 0 \iff e^x - 1 = 0 \iff e^x = 1 \iff e^x = e^0 \iff x = 0$$

$$f'(x) < 0 \iff x < 0$$

x	$-\infty$	0	$+\infty$
f'	-	0	+
f			

Επειδή: $\left\{ \begin{array}{l} \text{I) Η συνάρτηση } f \text{ είναι συνεχής στη θέση } x_0 = 0 \\ \text{II) } f'(x) < 0 \text{ για κάθε } x \in (0, 1) \\ \text{III) } f'(x) > 0 \text{ για κάθε } x \in (1, +\infty) \end{array} \right.$

Η συνάρτηση f έχει ελάχιστο στη θέση $x_0 = 0$ τον αριθμό :

$$f(0) = e^0 - 0 - 1 = 0$$

Οπότε για κάθε $x \in \mathbb{R}$ ισχύει :

$$f(x) \geq f(0) \iff e^x - x - 1 \geq 0 \iff e^x \geq x + 1$$

2.

Να αποδείξετε ότι για κάθε $x \in (0, +\infty)$: $\ln x \leq x - 1$

Θεωρώ τη συνάρτηση $f(x) = \ln x - x + 1$, $x \in (0, +\infty)$

$$f'(x) = (\ln x - x + 1)' = (\ln x)' - (x)' + (1)' = \frac{1}{x} - 1 - 0 = \frac{1}{x} - \frac{x}{x} =$$

$$= \frac{1-x}{x}$$



$$\left. \begin{array}{l} f'(x) > 0 \\ x > 0 \end{array} \right\} \Leftrightarrow \left. \begin{array}{l} \frac{1-x}{x} > 0 \\ x > 0 \end{array} \right\} \Leftrightarrow \left. \begin{array}{l} 1-x > 0 \\ x > 0 \end{array} \right\} \Leftrightarrow \left. \begin{array}{l} -x > -1 \\ x > 0 \end{array} \right\} \Leftrightarrow$$

$$\left. \begin{array}{l} \frac{-x}{-1} < \frac{-1}{-1} \\ x > 0 \end{array} \right\} \Leftrightarrow \left. \begin{array}{l} x < 1 \\ x > 0 \end{array} \right\} \Leftrightarrow 0 < x < 1$$

$$\left. \begin{array}{l} f'(x) = 0 \\ x > 0 \end{array} \right\} \Leftrightarrow \left. \begin{array}{l} \frac{1-x}{x} = 0 \\ x > 0 \end{array} \right\} \Leftrightarrow \left. \begin{array}{l} 1-x = 0 \\ x > 0 \end{array} \right\} \Leftrightarrow \left. \begin{array}{l} -x = -1 \\ x > 0 \end{array} \right\} \Leftrightarrow$$

$$\left. \begin{array}{l} \frac{-x}{-1} = \frac{-1}{-1} \\ x > 0 \end{array} \right\} \Leftrightarrow \left. \begin{array}{l} x = 1 \\ x > 0 \end{array} \right\} \Leftrightarrow x = 1$$

$$\left. \begin{array}{l} f'(x) < 0 \\ x > 0 \end{array} \right\} \Leftrightarrow x > 1$$

x	0	1	$+\infty$	
f'		+	0	-
f			0	

max

Επειδή: $\left\{ \begin{array}{l} \text{I) Η συνάρτηση } f \text{ είναι συνεχής στη θέση } x_0 = 1 \\ \text{II) } f'(x) > 0 \text{ για κάθε } x \in (0, 1) \\ \text{III) } f'(x) < 0 \text{ για κάθε } x \in (1, +\infty) \end{array} \right.$

Η συνάρτηση f έχει μέγιστο στη θέση $x_0 = 1$ τον αριθμό :
 $f(1) = \ln 1 - 1 + 1 = 0$

Οπότε για κάθε $x \in (0, +\infty)$ ισχύει :
 $f(x) \leq f(1) \Leftrightarrow \ln x - x + 1 \leq 0 \Leftrightarrow \ln x \leq x - 1$
 3.

Να αποδείξετε ότι : $2\sqrt{x} \geq 3 - \frac{1}{x}, x \in (0, +\infty)$

Θεωρώ τη συνάρτηση $f(x) = 2\sqrt{x} \geq 3 - \frac{1}{x}, x \in (0, +\infty)$

$$f'(x) = \left(2\sqrt{x} - 3 + \frac{1}{x}\right)' \stackrel{\substack{\sqrt{a}=a^{\frac{1}{2}}, a \geq 0 \\ a^{-1}=\frac{1}{a}, a \neq 0}}{=} \left(2x^{\frac{1}{2}} - 3 + x^{-1}\right)' \stackrel{(F(x)-G(x)+H(x))'=F'(x)-G'(x)+H'(x)}{=} \\ \stackrel{(\lambda F(x))'=\lambda F'(x)}{\left(2x^{\frac{1}{2}}\right)'} - \stackrel{(c)'=0, c: \text{σταθερά}}{(3)'} + \stackrel{(x^a)'=ax^{a-1}}{(x^{-1})}' = 2 \left(x^{\frac{1}{2}}\right)' - x^{-2} \stackrel{a^{-v}=\frac{1}{\alpha^v}, \alpha \neq 0}{=} 2 \frac{1}{2} x^{-\frac{1}{2}} - \frac{1}{x^2} = \frac{1}{x^{\frac{1}{2}}} - \frac{1}{x^2} \\ = \frac{1}{\sqrt{x}} - \frac{1}{x^2} = \frac{1 \cdot \sqrt{x}}{\sqrt{x} \sqrt{x}} - \frac{1}{x^2} = \frac{\sqrt{x}}{(\sqrt{x})^2} - \frac{1}{x^2} \stackrel{(\sqrt{\theta})^2=\theta \geq 0}{=} \frac{x\sqrt{x}}{xx} - \frac{1}{x^2} = \frac{x\sqrt{x}-1}{x^2}$$

$$\left\{ \begin{array}{l} f(x) > 0 \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \frac{x\sqrt{x}-1}{x^2} > 0 \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} x\sqrt{x}-1 > 0 \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} x\sqrt{x} > 1 \\ x > 0 \end{array} \right\} \stackrel{\substack{\text{Αν } \alpha, \beta > 0 \text{ τότε ισχύει} \\ \text{η ισοδυναμία:} \\ \alpha > \beta \Leftrightarrow \alpha^2 > \beta^2}}{\Leftrightarrow}$$

$$\left\{ \begin{array}{l} (x\sqrt{x})^2 > 1^2 \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} x^2 (\sqrt{x})^2 > 1 \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} x^2 \cdot x > 1 \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} x^3 > 1 \\ x > 0 \end{array} \right\} \stackrel{\substack{\text{Αν } \alpha, \beta > 0 \text{ τότε ισχύει} \\ \text{η ισοδυναμία:} \\ \alpha > \beta \Leftrightarrow \sqrt[3]{\alpha} > \sqrt[3]{\beta}}{\Leftrightarrow}$$

$$\Leftrightarrow \left\{ \begin{array}{l} \sqrt[3]{x^3} > \sqrt[3]{1} \\ x > 0 \end{array} \right\} \stackrel{\substack{\text{Αν } \alpha \geq 0 \text{ τότε ισχύει} \\ \sqrt[3]{a^3}=a}}{\Leftrightarrow} \left\{ \begin{array}{l} x > 1 \\ x > 0 \end{array} \right\} \Leftrightarrow x > 1$$

$$\left\{ \begin{array}{l} f(x) = 0 \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \frac{x\sqrt{x}-1}{x^2} = 0 \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} x\sqrt{x}-1 = 0 \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} x\sqrt{x} = 1 \\ x > 0 \end{array} \right\} \stackrel{\substack{\text{Αν } \alpha, \beta > 0 \text{ τότε ισχύει} \\ \text{η ισοδυναμία:} \\ \alpha = \beta \Leftrightarrow \alpha^2 = \beta^2}}{\Leftrightarrow}$$

$$\left\{ \begin{array}{l} (x\sqrt{x})^2 = 1^2 \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} x^2 (\sqrt{x})^2 = 1 \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} x^2 \cdot x = 1 \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} x^3 = 1 \\ x > 0 \end{array} \right\} \Leftrightarrow$$

Αν $\alpha, \beta > 0$ τότε ισχύει
η ισοδυναμία:
 $\alpha = \beta \Leftrightarrow \sqrt[3]{\alpha} = \sqrt[3]{\beta}$

$$\Leftrightarrow \left\{ \begin{array}{l} \sqrt[3]{x^3} = \sqrt[3]{1} \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} x = 1 \\ x > 0 \end{array} \right\} \Leftrightarrow x = 1$$

Αν $a \geq 0$ τότε ισχύει
 $\sqrt[3]{a^3} = a$

$$\left\{ \begin{array}{l} f(x) < 0 \\ x > 0 \end{array} \right\} \Leftrightarrow 0 < x < 1$$

x	0	1	$+\infty$
f'	-	0	+
f			

min

Επειδή: $\left\{ \begin{array}{l} \text{I) Η συνάρτηση } f \text{ είναι συνεχής στη θέση } x_0 = 1 \\ \text{II) } f'(x) < 0 \text{ για κάθε } x \in (0, 1) \\ \text{III) } f'(x) > 0 \text{ για κάθε } x \in (1, +\infty) \end{array} \right.$

Η συνάρτηση f έχει ελάχιστο στη θέση $x_0 = 1$ τον αριθμό :

$$f(1) = 2\sqrt{1} - 3 + \frac{1}{1} = 2 - 3 + 1 = 0$$

Οπότε για κάθε $x \in \mathbb{R}$ ισχύει :

$$f(x) \geq f(1) \Leftrightarrow 2\sqrt{x} - 3 + \frac{1}{x} \geq 0 \Leftrightarrow 2\sqrt{x} \geq 3 - \frac{1}{x}, x \in (0, +\infty)$$

4.

Να αποδείξετε ότι για κάθε $x \in (0, \frac{\pi}{2})$ ισχύει $\varepsilon\varphi x + \sigma\varphi x \geq 2$

$$(\sigma\varphi x)' = (\varepsilon\varphi^{-1}x)' \stackrel{(f^a)' = af^{a-1}f'}{=} -1\varepsilon\varphi^{-2}x(\varepsilon\varphi x)' = -\frac{\sigma\nu\nu^2x}{\eta\mu^2x} \cdot \frac{1}{\sigma\nu\nu^2x} = -\frac{1}{\eta\mu^2x}$$

Θεωρώ τη συνάρτηση $f(x) = \varepsilon\varphi x + \sigma\varphi x - 2$

$$\begin{aligned}
 f'(x) &= (\epsilon\phi x + \sigma\phi x - 2)' = (\epsilon\phi x)' + (\sigma\phi x)' - (2)' = \frac{1}{\sigma\upsilon\nu^2 x} - \frac{1}{\eta\mu^2 x} - 0 = \\
 &= \frac{\eta\mu^2 x}{\eta\mu^2 x \sigma\upsilon\nu^2 x} - \frac{\sigma\upsilon\nu^2 x}{\eta\mu^2 x \sigma\upsilon\nu^2 x} = \frac{\eta\mu^2 x - \sigma\upsilon\nu^2 x}{\eta\mu^2 x \sigma\upsilon\nu^2 x} = \frac{-(\sigma\upsilon\nu^2 x - \eta\mu^2 x)}{\eta\mu^2 x \sigma\upsilon\nu^2 x} = \\
 &= \frac{-\sigma\upsilon\nu 2x}{\eta\mu^2 x \sigma\upsilon\nu^2 x} = -\frac{\sigma\upsilon\nu 2x}{\eta\mu^2 x \sigma\upsilon\nu^2 x}
 \end{aligned}$$

$\sigma\upsilon\nu 2x = \sigma\upsilon\nu^2 x - \eta\mu^2 x$
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Θεωρώ τη συνάρτηση $g(x) = \sigma\upsilon\nu 2x$, $x \in (0, \frac{\pi}{2})$

$$g'(x) = (\sigma\upsilon\nu 2x)' = -\eta\mu 2x \quad (2x)' = 2 \Rightarrow -2\eta\mu 2x$$

$$\text{Έχω : } 0 < x < \frac{\pi}{2} \implies 2 \cdot 0 < 2x < 2 \cdot \frac{\pi}{2} \implies 0 < 2x < \pi \implies \eta\mu 2x > 0 \implies$$

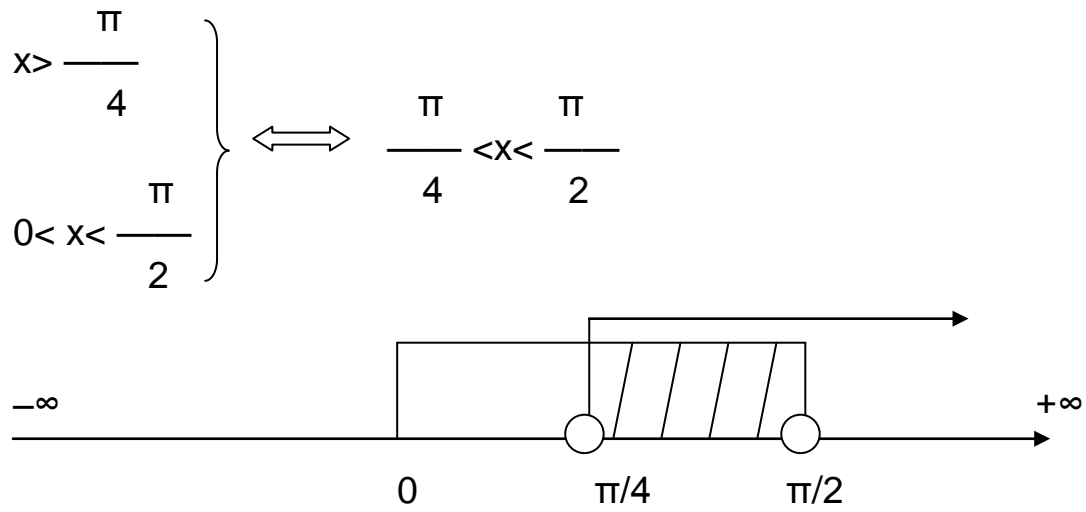
$$-2\eta\mu 2x < 0 \implies g'(x) < 0$$

Επειδή $g'(x) < 0$ για κάθε $x \in (0, \frac{\pi}{2})$ η g είναι γνησίως φθίνουσα

Οπότε η g είναι «1-1»

$$\left. \begin{array}{l} f'(x) > 0 \\ x \in (0, \frac{\pi}{2}) \end{array} \right\} \iff \left. \begin{array}{l} -\frac{\sigma\upsilon\nu 2x}{\eta\mu^2 x \sigma\upsilon\nu^2 x} > 0 \\ x \in (0, \frac{\pi}{2}) \end{array} \right\} \iff \left. \begin{array}{l} \sigma\upsilon\nu 2x < 0 \\ x \in (0, \frac{\pi}{2}) \end{array} \right\} \iff$$

$$\left. \begin{array}{l} g(x) < 0 \\ x \in (0, \frac{\pi}{2}) \end{array} \right\} \iff \left. \begin{array}{l} g(x) < g(\frac{\pi}{4}) \\ x \in (0, \frac{\pi}{2}) \end{array} \right\} \iff \text{Η συνάρτηση } g \text{ είναι γνησίως φθίνουσα}$$





$$\left. \begin{array}{l} f'(x)=0 \\ x \in (0, \frac{\pi}{2}) \end{array} \right\} \iff \left. \begin{array}{l} -\frac{\sin 2x}{\eta \mu^2 x \cos^2 x} = 0 \\ x \in (0, \frac{\pi}{2}) \end{array} \right\} \iff \left. \begin{array}{l} \sin 2x = 0 \\ x \in (0, \frac{\pi}{2}) \end{array} \right\} \iff$$

$$\left. \begin{array}{l} g(x)=0 \\ x \in (0, \frac{\pi}{2}) \end{array} \right\} \iff \left. \begin{array}{l} g(x) = g(\frac{\pi}{4}) \\ x \in (0, \frac{\pi}{2}) \end{array} \right\} \iff \begin{array}{l} \text{Η συνάρτηση } g \\ \text{είναι «1-1»} \end{array}$$

$$\left. \begin{array}{l} x = \frac{\pi}{4} \\ 0 < x < \frac{\pi}{2} \end{array} \right\} \iff x = \frac{\pi}{4}$$

$$\left. \begin{array}{l} f'(x) < 0 \\ x \in (0, \frac{\pi}{2}) \end{array} \right\} \iff 0 < x < \frac{\pi}{4}$$

	2		
x	0	π/4	π/2
f'	-	0	+
f			

min

Επειδή: $\left\{ \begin{array}{l} \text{I) Η συνάρτηση } f \text{ είναι συνεχής στη θέση } x_0 = \pi/2 \\ \text{II) } f'(x) < 0 \text{ για κάθε } x \in (0, \pi/4) \\ \text{III) } f'(x) > 0 \text{ για κάθε } x \in (\pi/4, \pi/2) \end{array} \right.$

Η συνάρτηση f έχει ελάχιστο στη θέση $x_0 = \pi/4$ τον αριθμό :

$$f\left(\frac{\pi}{4}\right) = \varepsilon\varphi\frac{\pi}{4} + \sigma\varphi\frac{\pi}{4} - 2 = 1 + 1 - 2 = 0$$

Οπότε για κάθε $x \in (0, \frac{\pi}{4})$ ισχύει :

$$f(x) \geq f\left(\frac{\pi}{4}\right) \Leftrightarrow \varepsilon\varphi x + \sigma\varphi x - 2 \geq 0 \Leftrightarrow \varepsilon\varphi x + \sigma\varphi x \geq 2$$

5.

Για κάθε $x \in (0, 1)$ ισχύει $x^x(1-x)^{1-x} \geq \frac{1}{2}$

Θεωρώ τη συνάρτηση $f(x) = x^x(1-x)^{1-x} - \frac{1}{2}$, $x \in (0, 1)$

$$f'(x) = (x^x(1-x)^{1-x} - \frac{1}{2})' = (x^x(1-x)^{1-x})' - (\frac{1}{2})' =$$

$$= (x^x)'(1-x)^{1-x} + x^x((1-x)^{1-x})' = (e^{\ln x^x})'(1-x)^{1-x} + x^x(e^{\ln(1-x)^{1-x}})' =$$

$$= (e^{x \ln x})'(1-x)^{1-x} + x^x(e^{(1-x) \ln(1-x)})' =$$

$$\begin{aligned}
&= e^{x \ln x} (x \ln x)' (1-x)^{1-x} + x^x e^{(1-x) \ln(1-x)} ((1-x) \ln(1-x))' = \\
&= e^{\ln x^x} [(x)' \ln x + x (\ln x)'] (1-x)^{1-x} + x^x e^{\ln(1-x)^{1-x}} [(1-x)' \ln(1-x) + (1-x) (\ln 1-x)'] = \\
&= x^x (1-x)^{1-x} \left(\ln x + x \frac{1}{x} \right) + x^x (1-x)^{1-x} \left(-\ln(1-x) + (1-x) \frac{(1-x)'}{1-x} \right) \\
&= x^x (1-x)^{1-x} (\ln x + 1 - \ln(1-x) - 1) = x^x (1-x)^{1-x} \ln \frac{x}{1-x}
\end{aligned}$$

$(e^f)' = e^f \cdot f'$
$\theta = \ln e^\theta, \theta > 0$
$x \cdot \ln \theta = \ln \theta^x, \theta > 0$
$\ln \theta_1 - \ln \theta_2 = \ln \frac{\theta_1}{\theta_2} \quad \mu \varepsilon \theta_1, \theta_2 > 0$

Για κάθε $x \in (0, 1)$ θα ισχύει :

$$\left. \begin{array}{l} x > 0 \\ x < 1 \end{array} \right\} \Rightarrow \left. \begin{array}{l} x > 0 \\ 1 > x \end{array} \right\} \Rightarrow \left. \begin{array}{l} x > 0 \\ 1 - x > 0 \end{array} \right\} \Rightarrow \left. \begin{array}{l} x^x > 0 \\ (1-x)^{1-x} > 0 \end{array} \right\} \Rightarrow$$

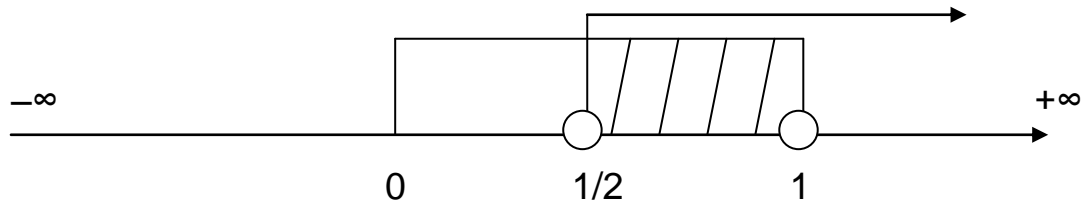
$$\Rightarrow x^x (1-x)^{1-x} > 0$$

$$\left. \begin{array}{l} f'(x) > 0 \\ 0 < x < 1 \end{array} \right\} \Leftrightarrow \left. \begin{array}{l} \ln \frac{x}{1-x} > 0 \\ 0 < x < 1 \end{array} \right\} \Leftrightarrow \left. \begin{array}{l} \ln \frac{x}{1-x} > \ln 1 \\ 0 < x < 1 \end{array} \right\} \Leftrightarrow$$

$$\left. \begin{array}{l} \frac{x}{1-x} > 1 \\ 0 < x < 1 \end{array} \right\} \Leftrightarrow \left. \begin{array}{l} \frac{x}{1-x} (1-x) > 1(1-x) \\ 0 < x < 1 \text{ (Γιατί : } 1 > x \text{ ή } 1-x > 0) \end{array} \right\} \Leftrightarrow$$

$$\left. \begin{array}{l} x > 1-x \\ 0 < x < 1 \end{array} \right\} \Leftrightarrow \left. \begin{array}{l} x+x > 1 \\ 0 < x < 1 \end{array} \right\} \Leftrightarrow \left. \begin{array}{l} 2x > 1 \\ 0 < x < 1 \end{array} \right\} \Leftrightarrow$$

$$\left. \begin{array}{l} \frac{2x}{2} > \frac{1}{2} \\ 0 < x < 1 \end{array} \right\} \iff \left. \begin{array}{l} x > \frac{1}{2} \\ 0 < x < 1 \end{array} \right\} \iff \frac{1}{2} < x < 1$$





$$\left. \begin{array}{l} f'(x) = 0 \\ 0 < x < 1 \end{array} \right\} \iff \left. \begin{array}{l} \ln \frac{x}{1-x} = 0 \\ 0 < x < 1 \end{array} \right\} \iff \left. \begin{array}{l} \ln \frac{x}{1-x} = \ln 1 \\ 0 < x < 1 \end{array} \right\} \iff$$

$$\left. \begin{array}{l} \frac{x}{1-x} = 1 \\ 0 < x < 1 \end{array} \right\} \iff \left. \begin{array}{l} x = 1-x \\ 0 < x < 1 \end{array} \right\} \iff \left. \begin{array}{l} x+x = 1 \\ 0 < x < 1 \end{array} \right\} \iff \left. \begin{array}{l} 2x = 1 \\ 0 < x < 1 \end{array} \right\} \iff$$

$$\left. \begin{array}{l} \frac{2x}{2} = \frac{1}{2} \\ 0 < x < 1 \end{array} \right\} \iff \left. \begin{array}{l} x = \frac{1}{2} \\ 0 < x < 1 \end{array} \right\} \iff x = \frac{1}{2}$$

$$\left. \begin{array}{l} f'(x) > 0 \\ 0 < x < 1 \end{array} \right\} \iff 0 < x < \frac{1}{2}$$

x	0	1/2	1
f'	-	0	+
f			

min

Επειδή: $\left\{ \begin{array}{l} \text{I) Η συνάρτηση } f \text{ είναι συνεχής στη θέση } x_0 = 1/2 \\ \text{II) } f'(x) < 0 \text{ για κάθε } x \in (0, 1/2) \\ \text{III) } f'(x) > 0 \text{ για κάθε } x \in (1/2, 1) \end{array} \right.$

Η συνάρτηση f έχει ελάχιστο στη θέση $x_0 = 1/2$ τον αριθμό :

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{1/2} \left(1 - \frac{1}{2}\right)^{1-1/2} - \frac{1}{2} = \left(\frac{1}{2}\right)^{1/2} \left(\frac{1}{2}\right)^{1/2} - \frac{1}{2} = 0$$

Οπότε για κάθε $x \in (0, 1)$ ισχύει :

$$f(x) \geq f\left(\frac{1}{2}\right) \Leftrightarrow x^x(1-x)^{1-x} - \frac{1}{2} \geq 0 \Leftrightarrow x^x(1-x)^{1-x} \geq \frac{1}{2}$$

5.

Να αποδείξετε ότι: $x e^{-x^2} \leq \frac{1}{\sqrt{2e}}, x \geq 0$

Θεωρώ τη συνάρτηση $f(x) = x e^{-x^2} - \frac{1}{\sqrt{2e}}, x \geq 0$

$$\begin{aligned} f'(x) &= \left(x e^{-x^2} - \frac{1}{\sqrt{2e}} \right)' = \left(x e^{-x^2} \right)' - \left(\frac{1}{\sqrt{2e}} \right)' = (x)' e^{-x^2} + x (e^{-x^2})' = \\ &= e^{-x^2} + x (-x^2)' e^{-x^2} = e^{-x^2} - 2x^2 e^{-x^2} = e^{-x^2} (1 - 2x^2) \end{aligned}$$

$$\left\{ \begin{array}{l} f'(x) > 0 \\ x \geq 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} e^{-x^2} (1 - 2x^2) > 0 \\ x \geq 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} e^{-x^2} > 0 \\ 1 - 2x^2 > 0 \\ x \geq 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} -2x^2 > -1 \\ x \geq 0 \end{array} \right\}$$

Όταν διαιρώ και τα
δύο μέλη μιας
ανίσωσης με αρνητικό
αριθμό προκύπτει
ετερόστροφη ανίσωση

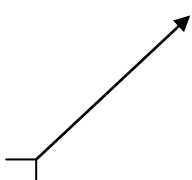
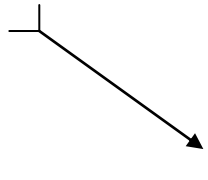
$$\Leftrightarrow \begin{cases} \frac{-2x^2}{-2} < \frac{-1}{-2} \\ x \geq 0 \end{cases} \Leftrightarrow \begin{cases} x^2 < \frac{1}{2} \\ x \geq 0 \end{cases} \Leftrightarrow \begin{cases} \sqrt{x^2} < \sqrt{\frac{1}{2}} \\ x \geq 0 \end{cases} \Leftrightarrow \begin{cases} \sqrt{x^2} = |x| \\ |x| < \frac{1}{\sqrt{2}} \\ x \geq 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} |x| < \frac{1 \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} \\ x \geq 0 \end{cases} \Leftrightarrow \begin{cases} |x| < \frac{\sqrt{2}}{2} \\ x \geq 0 \end{cases} \begin{matrix} |x| < \theta \Leftrightarrow -\theta < x < \theta, \theta > 0 \\ \Leftrightarrow \end{matrix} \begin{cases} -\frac{\sqrt{2}}{2} < x < \frac{\sqrt{2}}{2} \\ x \geq 0 \end{cases} \Leftrightarrow 0 \leq x < \frac{\sqrt{2}}{2}$$

$$\begin{cases} f'(x) = 0 \\ x \geq 0 \end{cases} \Leftrightarrow \begin{cases} e^{-x^2} (1 - 2x^2) = 0 \\ x \geq 0 \end{cases} \Leftrightarrow \begin{matrix} e^{-x^2} > 0 \\ \begin{cases} 1 - 2x^2 = 0 \\ x \geq 0 \end{cases} \end{matrix} \Leftrightarrow \begin{cases} -2x^2 = -1 \\ x \geq 0 \end{cases}$$

$$\begin{cases} x^2 = \frac{-1}{-2} \\ x \geq 0 \end{cases} \begin{matrix} x^2 = \theta \Leftrightarrow x = \pm \sqrt{\theta}, \theta \geq 0 \\ \Leftrightarrow \end{matrix} \begin{cases} x = \pm \sqrt{\frac{1}{2}} \\ x \geq 0 \end{cases} \Leftrightarrow \begin{cases} x = \pm \frac{\sqrt{2}}{2} \\ x \geq 0 \end{cases} \Leftrightarrow x = \frac{\sqrt{2}}{2}$$

$$\begin{cases} f'(x) < 0 \\ x \geq 0 \end{cases} \Leftrightarrow x > \frac{\sqrt{2}}{2}$$

x	0	$\frac{\sqrt{2}}{2}$	$+\infty$
f'	$+$	0	$-$
f			

max

$$\text{Οπότε: } \left\{ \begin{array}{l} \text{(I) Η } f \text{ είναι συνεχής στην θέση } x_0 = \frac{\sqrt{2}}{2} \\ \text{(II) } f'(x) > 0, x \in \left[0, \frac{\sqrt{2}}{2} \right) \\ \text{(III) } f'(x) < 0, x \in \left(\frac{\sqrt{2}}{2}, +\infty \right) \end{array} \right\}$$

Άρα η συνάρτηση f έχει μέγιστο στην θέση $x_0 = \frac{\sqrt{2}}{2}$ τον αριθμό:

$$f(x_0) = f\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2} e^{-\left(\frac{\sqrt{2}}{2}\right)^2} - \frac{1}{\sqrt{2e}} = \frac{\sqrt{2}}{2} e^{-\frac{1}{2}} - \frac{1}{\sqrt{2e}} = \frac{\sqrt{2}}{2} \frac{1}{e^{\frac{1}{2}}} - \frac{1}{\sqrt{2e}}$$

$$\begin{aligned} & \sqrt{a} = a^{\frac{1}{2}}, a \geq 0 \\ & 2 = (\sqrt{2})^2 \\ & = \frac{\cancel{\sqrt{2}}}{(\sqrt{2})^{\cancel{2}} \sqrt{e}} - \frac{1}{\sqrt{2e}} = \frac{1}{\sqrt{2}\sqrt{e}} - \frac{1}{\sqrt{2e}} \stackrel{\sqrt{a}\sqrt{b} = \sqrt{ab}, \alpha, \beta \geq 0}{=} \frac{1}{\sqrt{2e}} - \frac{1}{\sqrt{2e}} = 0 \end{aligned}$$

Συνεπώς για κάθε $x \geq 0$ θα ισχύει:

$$\left\{ \begin{array}{l} f(x) \leq f\left(\frac{\sqrt{2}}{2}\right) \\ x \geq 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} xe^{-x^2} - \frac{1}{\sqrt{2e}} \leq 0 \\ x \geq 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} xe^{-x^2} \leq \frac{1}{\sqrt{2e}} \\ x \geq 0 \end{array} \right\}$$

6.

Να αποδείξετε ότι: $x^3 \ln x \geq -\frac{1}{3e}, x > 0$

Θεωρώ την συνάρτηση $f(x) = x^3 \ln x + \frac{1}{3e}, x > 0$

$$f'(x) = \left(x^3 \ln x + \frac{1}{3e} \right)' \stackrel{(F(x)-G(x))' = F'(x)+G'(x)}{=} (x^3 \ln x)' + \left(\frac{1}{3e} \right)'$$

$$\begin{aligned}
 (F(x)G(x))' &= F'(x)G(x) + F(x)G'(x) & (x^a)' &= ax^{a-1} \\
 (c^x)' &= 0, c: \text{σταθερά} & (\ln x)' &= \frac{1}{x}, x > 0 \\
 &= (x^3)' \ln x + x^3 (\ln x)' & &= 3x^2 \ln x + x^3 \frac{1}{x} =
 \end{aligned}$$

$$3x^2 \ln x + x^2 = x^2 (3 \ln x + 1)$$

$$\left\{ \begin{array}{l} f'(x) > 0 \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} x^2 (3 \ln x + 1) > 0 \\ x > 0 \end{array} \right\} \stackrel{\text{Αν } x \neq 0 \text{ έχω } x^2 > 0}{\Leftrightarrow} \left\{ \begin{array}{l} 3 \ln x + 1 > 0 \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} 3 \ln x > -1 \\ x > 0 \end{array} \right\}$$

$$\Leftrightarrow \left\{ \begin{array}{l} \ln x > -\frac{1}{3} \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \ln x > -\frac{1}{3} \cdot 1 \\ x > 0 \end{array} \right\} \stackrel{\ln e = 1}{\Leftrightarrow} \left\{ \begin{array}{l} \ln x > -\frac{1}{3} \cdot \ln e \\ x > 0 \end{array} \right\} \stackrel{x \ln \theta = \ln \theta^x, \theta > 0}{\Leftrightarrow} \left\{ \begin{array}{l} \ln x > \ln e^{-\frac{1}{3}} \\ x > 0 \end{array} \right\}$$

Αν $\theta_1, \theta_2 > 0$ τότε ισχύει η ισοδυναμία:
 $\ln \theta_1 > \ln \theta_2 \Leftrightarrow \theta_1 > \theta_2$

$$\Leftrightarrow \left\{ \begin{array}{l} x > e^{-\frac{1}{3}} \\ x > 0 \end{array} \right\} \Leftrightarrow x > e^{-\frac{1}{3}}$$

$$\left\{ \begin{array}{l} f'(x) = 0 \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} x^2 (3 \ln x + 1) = 0 \\ x > 0 \end{array} \right\} \stackrel{\text{Αν } x \neq 0 \text{ έχω } x^2 > 0}{\Leftrightarrow} \left\{ \begin{array}{l} 3 \ln x + 1 = 0 \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} 3 \ln x = -1 \\ x > 0 \end{array} \right\}$$

$$\Leftrightarrow \left\{ \begin{array}{l} \ln x = -\frac{1}{3} \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \ln x = -\frac{1}{3} \cdot 1 \\ x > 0 \end{array} \right\} \stackrel{\ln e = 1}{\Leftrightarrow} \left\{ \begin{array}{l} \ln x = -\frac{1}{3} \cdot \ln e \\ x > 0 \end{array} \right\} \stackrel{x \ln \theta = \ln \theta^x, \theta > 0}{\Leftrightarrow} \left\{ \begin{array}{l} \ln x = \ln e^{-\frac{1}{3}} \\ x > 0 \end{array} \right\}$$

Αν $\theta_1, \theta_2 > 0$ τότε ισχύει η ισοδυναμία:
 $\ln \theta_1 > \ln \theta_2 \Leftrightarrow \theta_1 > \theta_2$

$$\Leftrightarrow \left\{ \begin{array}{l} x = e^{-\frac{1}{3}} \\ x > 0 \end{array} \right\} \Leftrightarrow x = e^{-\frac{1}{3}}$$

$$\left\{ \begin{array}{l} f'(x) < 0 \\ x > 0 \end{array} \right\} \Leftrightarrow 0 < x < e^{-\frac{1}{3}}$$

x	0	$e^{\frac{1}{3}}$	$+\infty$
f'	$-$	0	$+$
f			

min

$$\text{Οπότε: } \left\{ \begin{array}{l} \text{(I) Η } f \text{ είναι συνεχής στην θέση } x_0 = e^{\frac{1}{3}} \\ \text{(II) } f'(x) < 0, x \in \left(0, e^{\frac{1}{3}}\right) \\ \text{(III) } f'(x) > 0, x \in \left(e^{\frac{1}{3}}, +\infty\right) \end{array} \right\}$$

Άρα η συνάρτηση f έχει ελάχιστο στην θέση $x_0 = e^{\frac{1}{3}}$ τον αριθμό:

$$\begin{aligned} f(x_0) &= f\left(e^{\frac{1}{3}}\right) = \left(e^{\frac{1}{3}}\right)^3 \ln e^{\frac{1}{3}} + \frac{1}{3e} \stackrel{(a^u)^v = a^{uv}}{=} e^{-3 \cdot \frac{1}{3}} \left(-\frac{1}{3}\right) \ln e + \frac{1}{3e} \stackrel{x \ln \theta = \ln \theta^x, \theta > 0}{=} \\ &= e^{-1} \left(-\frac{1}{3}\right) \ln e + \frac{1}{3e} = -\frac{e^{-1}}{3} + \frac{1}{3e} \stackrel{a^{-1} = \frac{1}{a}, a \neq 0}{=} -\frac{1}{3e} + \frac{1}{3e} = 0 \end{aligned}$$

Συνεπώς για κάθε $x > 0$ θα ισχύει:

$$\left\{ \begin{array}{l} f(x) \geq f\left(e^{-\frac{1}{3}}\right) \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} x^3 \ln x + \frac{1}{3e} \geq 0 \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} x^3 \ln x \geq -\frac{1}{3e} \\ x > 0 \end{array} \right\}$$

7.

Να αποδείξετε ότι: $\frac{\ln x}{x} \geq \frac{1}{e}, x > 0$

Θεωρώ την συνάρτηση: $f(x) = \frac{\ln x}{x} - \frac{1}{e}, x > 0$

$$f'(x) = \left(\frac{\ln x}{x} - \frac{1}{e} \right)' \stackrel{(F(x)-G(x))' = F'(x)-G'(x)}{=} \left(\frac{\ln x}{x} \right)' - \left(\frac{1}{e} \right)'$$

$$\begin{aligned} \left(\frac{F(x)}{G(x)} \right)' &= \frac{F'(x)G(x) - F(x)G'(x)}{G^2(x)} \\ \stackrel{(c)=0, c:\Sigma\tauαθερά}{=} & \frac{(\ln x)' x - \ln x (x)'}{x^2} \stackrel{(\ln x)' = \frac{1}{x}, x > 0}{=} \frac{\frac{1}{x} x - \ln x}{x^2} = \frac{1 - \ln x}{x^2} \end{aligned}$$

$$\left\{ \begin{array}{l} f'(x) > 0 \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \frac{1 - \ln x}{x^2} > 0 \\ x > 0 \end{array} \right\} \stackrel{\text{Αν } x \neq 0 \text{ θα έχω } x^2 > 0}{\Leftrightarrow} \left\{ \begin{array}{l} 1 - \ln x > 0 \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} -\ln x > -1 \\ x > 0 \end{array} \right\}$$

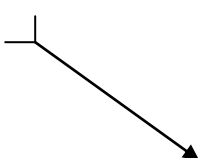
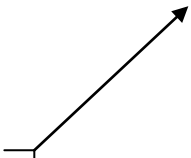
Όταν διαιρώ και τα δυο μέλη μιας ανίσωσης με αρνητικό αριθμό προκύπτει ετερόστροφη ανίσωση

$$\Leftrightarrow \left\{ \begin{array}{l} \frac{-\ln x}{-1} < \frac{-1}{-1} \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \ln x < 1 \\ x > 0 \end{array} \right\} \stackrel{\ln e = 1}{\Leftrightarrow} \left\{ \begin{array}{l} \ln x < \ln e \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} x < e \\ x > 0 \end{array} \right\} \Leftrightarrow 0 < x < e$$

$$\left\{ \begin{array}{l} f'(x) = 0 \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \frac{1 - \ln x}{x^2} = 0 \\ x > 0 \end{array} \right\} \stackrel{\text{Αν } x \neq 0 \text{ θα έχω } x^2 > 0}{\Leftrightarrow} \left\{ \begin{array}{l} 1 - \ln x = 0 \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \ln x = 1 \\ x > 0 \end{array} \right\}$$

$$\stackrel{\ln e = 1}{\Leftrightarrow} \left\{ \begin{array}{l} \ln x = \ln e \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} x = e \\ x > 0 \end{array} \right\} \Leftrightarrow x = e$$

$$\left\{ \begin{array}{l} f'(x) < 0 \\ x > 0 \end{array} \right\} \Leftrightarrow x > e$$

x	0	e	$+\infty$
f'	-	0	+
f			

min

$$\text{Οπότε: } \left\{ \begin{array}{l} \text{(I) Η } f \text{ είναι συνεχής στην θέση } x_0 = e \\ \text{(II) } f'(x) < 0, x \in (0, e) \\ \text{(III) } f'(x) > 0, x \in (e, +\infty) \end{array} \right\}$$

Άρα η συνάρτηση f έχει ελάχιστο στην θέση $x_0 = e$ τον αριθμό:

$$f(x_0) = f(e) = \frac{\ln e}{e} - \frac{1}{e} = \frac{1}{e} - \frac{1}{e} = 0$$

Συνεπώς για κάθε $x > 0$ θα ισχύει:

$$\left\{ \begin{array}{l} f(x) \geq f(e) \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \frac{\ln x}{x} - \frac{1}{e} \geq 0 \\ x > 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \frac{\ln x}{x} \geq \frac{1}{e} \\ x > 0 \end{array} \right\}$$

ΑΣΚΗΣΕΙΣ

1.

Να αποδείξετε ότι για κάθε $x \in \mathbb{R}$ ισχύει: $xe^x + 1 \geq e^x$

2.

Να αποδείξετε ότι για κάθε $x \in (0, +\infty)$: $x - x \ln x \leq 1$

3.

Να αποδείξετε ότι: $xe^{\frac{1}{x^2}} \geq \sqrt{2e}, x > 0$

4.

Να αποδείξετε ότι: $x \ln x \geq -\frac{1}{e}, x > 0$

5.

Να αποδείξετε ότι για κάθε $x \in \mathbb{R}$ ισχύει: $xe^{-x+1} \leq 1$