ΟΡΙΑ ΣΤΟ ΑΠΕΙΡΟ ΟΤΑΝ $x \to x_0$

$$\text{Αν}: \left\{ \begin{array}{l} (\mathrm{I})\ f(x) > 0 \text{ για κάθε } x \in (\alpha, x_0) \cup (x_0, \beta) \\ (\mathrm{II})\ \lim\limits_{x \to x_0} f(x) = 0 \end{array} \right\}$$ $$\text{Τότε}: \lim_{x \to x_0} \frac{1}{f(x)} = +\infty$$
$$\text{Αν}: \left\{ \begin{array}{l} (\mathrm{I})\ f(x) < 0 \text{ για κάθε } x \in (\alpha, x_0) \cup (x_0, \beta) \\ (\mathrm{II})\ \lim\limits_{x \to x_0} f(x) = 0 \end{array} \right\}$$ $$\text{Τότε}: \lim_{x \to x_0} \frac{1}{f(x)} = -\infty$$

ΕΠΙΤΡΕΠΤΕΣ ΠΡΑΞΕΙΣ
$(+\infty) + (+\infty) = +\infty$
$(-\infty) + (-\infty) = -\infty$
$(+\infty) + \alpha = \alpha + (+\infty) = +\infty, \alpha \in \mathbb{R}$
$(-\infty) + \alpha = \alpha + (-\infty) = -\infty, \alpha \in \mathbb{R}$
$(+\infty) \bullet \alpha = \alpha \bullet (+\infty) = +\infty, \alpha > 0$
$(+\infty) \bullet \alpha = \alpha \bullet (+\infty) = -\infty, \alpha < 0$
$(-\infty) \bullet \alpha = \alpha \bullet (-\infty) = -\infty, \alpha > 0$
$(-\infty) \bullet \alpha = \alpha \bullet (-\infty) = +\infty, \alpha < 0$
$(+\infty) \bullet (+\infty) = +\infty$
$(-\infty) \bullet (-\infty) = +\infty$
$(-\infty) \bullet (+\infty) = (+\infty) \bullet (-\infty) = -\infty$

ΜΗ ΕΠΙΤΡΕΠΤΕΣ ΠΡΑΞΕΙΣ
$(+\infty) - (+\infty)$
$(-\infty) - (-\infty)$
$(+\infty) 0$
$0 (+\infty)$
$(-\infty) 0$

$0(+\infty)$
$\dfrac{+\infty}{+\infty}$
$\dfrac{-\infty}{-\infty}$
$\dfrac{+\infty}{-\infty}$
$\dfrac{-\infty}{+\infty}$
0^0
∞^0
∞^∞
1^∞

ΧΡΗΣΙΜΑ ΟΡΙΑ
$\lim\limits_{x\to x_0}\dfrac{1}{(x-x_0)^{2\nu}}=+\infty, \nu\in\mathbb{N}^*$
$\lim\limits_{x\to x_0^-}\dfrac{1}{(x-x_0)^{2\nu+1}}=-\infty, \nu\in\mathbb{N}^*$
$\lim\limits_{x\to x_0^+}\dfrac{1}{(x-x_0)^{2\nu+1}}=+\infty, \nu\in\mathbb{N}^*$
$\lim\limits_{x\to x_0^+}\dfrac{1}{\lvert x-x_0\rvert^{\nu}}=+\infty, \nu\in\mathbb{N}^*$

<u>ΠΑΡΑΔΕΙΓΜΑΤΑ</u>

1.

Να βρεθεί το όριο $\lim\limits_{x\to 0}\dfrac{x+1}{\sigma\upsilon\nu x-1}$

$$\text{Έχω}:\left\{\begin{matrix}\sigma\upsilon\nu x<1\\ x\in\left(-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right)\end{matrix}\right\}\Leftrightarrow\left\{\begin{matrix}\sigma\upsilon\nu x-1<0\\ x\in\left(-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right)\end{matrix}\right\}$$

$$\acute{E}\chi\omega:\left\{\begin{array}{l}(\mathrm{I})\,\sigma\upsilon\nu x-1<0,\ x\in\left(-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right)\\(\mathrm{II})\lim\limits_{x\to 0}(\sigma\upsilon\nu x-1)=0\end{array}\right\}$$

$$\mathrm{O}\pi\acute{o}\tau\varepsilon:\lim_{x\to 0}\frac{1}{\sigma\upsilon\nu x-1}=-\infty$$

$$\lim_{x\to 0}\frac{x+1}{\sigma\upsilon\nu x-1}=\lim_{x\to 0}(x+1)\frac{1}{\sigma\upsilon\nu x-1}=\lim_{x\to 0}(x+1)\lim_{x\to 0}\frac{1}{\sigma\upsilon\nu x-1}=1(-\infty)=-\infty$$

2.

$\mathrm{N}\alpha\ \beta\rho\varepsilon\theta\varepsilon\acute{\iota}\ \tau o\ \acute{o}\rho\iota o\ \lim\limits_{x\to\frac{\pi}{2}}\frac{x+7}{\eta\mu x-1}$

$$\acute{E}\chi\omega:\left\{\begin{array}{l}\eta\mu x<1\\x\in\left(\pi,\frac{\pi}{2}\right)\cup\left(0,\frac{\pi}{2}\right)\end{array}\right\}\Rightarrow\left\{\begin{array}{l}\eta\mu x-1<0\\x\in\left(0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right)\end{array}\right\}$$

$$\acute{E}\chi\omega:\left\{\begin{array}{l}(\mathrm{I})\,\eta\mu x-1<0,\ x\in\left(0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right)\\(\mathrm{II})\lim\limits_{x\to\frac{\pi}{2}}(\eta\mu x-1)=0\end{array}\right\}$$

$$\mathrm{O}\pi\acute{o}\tau\varepsilon:\lim_{x\to 0}\frac{1}{\eta\mu x-1}=-\infty$$

$$\lim_{x\to\frac{\pi}{2}}\frac{x+7}{\eta\mu x-1}=\lim_{x\to\frac{\pi}{2}}(x+7)\frac{1}{\eta\mu x-1}=\lim_{x\to\frac{\pi}{2}}(x+7)\lim_{x\to\frac{\pi}{2}}\frac{1}{\eta\mu x-1}=\left(\frac{\pi}{2}+7\right)(-\infty)=-\infty$$

3.

$\mathrm{N}\alpha\ \beta\rho\varepsilon\theta\varepsilon\acute{\iota}\ \tau o\ \acute{o}\rho\iota o\ \lim\limits_{x\to 0}\frac{x+8}{x^4-x^2}$

$$\lim_{x\to 0}\frac{x+8}{x^4-x^2}=\lim_{x\to 0}\frac{x+8}{x^2\left(x^2-1\right)}=\lim_{x\to 0}\frac{1}{x^2}\frac{x+8}{x^2-1}=\lim_{x\to 0}\frac{1}{x^2}\lim_{x\to 0}\frac{x+8}{x^2-1}=(+\infty)(-8)=-\infty$$

4.

$\mathrm{N}\alpha\ \beta\rho\varepsilon\theta\varepsilon\acute{\iota}\ \tau o\ \acute{o}\rho\iota o\left(\mathrm{A}\nu\ \upsilon\pi\acute{\alpha}\rho\chi\varepsilon\iota\right)\lim\limits_{x\to 0}\frac{x+9}{x^6-x^3}$

$$\Theta\varepsilon\omega\rho\acute{\omega}\ \tau\eta\nu\ \sigma\upsilon\nu\acute{\alpha}\rho\tau\eta\sigma\eta\ f(x)=\frac{x+9}{x^6-x^3},\ x\in(-\infty,0)\cup(0,1)\cup(1,+\infty)$$

$$\mathrm{A}\nu\ x\in(-\infty,0)\cup(0,1)\cup(1,+\infty)\ \theta\alpha\ \acute{\varepsilon}\chi\omega:$$

$$f\left(x\right)=\frac{x+9}{x^6-x^3}=\frac{x+9}{x^3\left(x^3-1\right)}=\frac{1}{x^3}\frac{x+9}{x^3-1}$$

$$\lim_{x\to0^+} f\left(x\right)=\lim_{x\to0^+}\frac{1}{x^3}\frac{x+9}{x^3-1}=\lim_{x\to0^+}\frac{1}{x^3}\lim_{x\to0^+}\frac{x+9}{x^3-1}=\left(+\infty\right)\left(-9\right)=-\infty$$

$$\lim_{x\to0^-} f\left(x\right)=\lim_{x\to0^-}\frac{1}{x^3}\frac{x+9}{x^3-1}=\lim_{x\to0^-}\frac{1}{x^3}\lim_{x\to0^-}\frac{x+9}{x^3-1}=\left(-\infty\right)\left(-9\right)=+\infty$$

$\mathrm{E}\pi\varepsilon\iota\delta\acute{\eta}\ \lim_{x\to0^+} f\left(x\right)\neq\lim_{x\to0^-} f\left(x\right)\ \delta\varepsilon\nu\ \upsilon\pi\acute{\alpha}\rho\chi\varepsilon\iota\ \tau o\ \acute{o}\rho\iota o\ \tau\eta\varsigma\ f\ \sigma\tau o\ \sigma\eta\mu\varepsilon\acute{\iota}o\ x_0=0$

5.

$$\boxed{\mathrm{N}\alpha\ \beta\rho\varepsilon\theta\varepsilon\acute{\iota}\ \tau o\ \acute{o}\rho\iota o\ \lim_{x\to4}\frac{x+11}{\left|x-4\right|+9\left(x-4\right)^2}}$$

$$\lim_{x\to4}\frac{x+11}{\left|x-4\right|+9\left(x-4\right)^2}\overset{|\alpha|^2=\alpha^2}{=}\lim_{x\to4}\frac{x+11}{\left|x-4\right|+9\left|x-4\right|^2}=\lim_{x\to4}\frac{x+11}{\left|x-4\right|\left[1+9\left|x-4\right|\right]}=$$

$$=\lim_{x\to4}\frac{1}{\left|x-4\right|}\lim_{x\to4}\frac{x+11}{1+9\left|x-4\right|}=\left(+\infty\right)\bullet15=+\infty$$

6.

$$\boxed{\mathrm{N}\alpha\ \beta\rho\varepsilon\theta\varepsilon\acute{\iota}\ \tau o\ \acute{o}\rho\iota o\ \lim_{x\to1}\frac{x+13}{x\sqrt{x}-x-\sqrt{x}+1}}$$

$$\lim_{x\to1}\frac{x+13}{x\sqrt{x}-x-\sqrt{x}+1}=\lim_{x\to1}\frac{x+13}{x\sqrt{x}-\sqrt{x}-x+1}=\lim_{x\to1}\frac{x+13}{\sqrt{x}\left(x-1\right)-\left(x-1\right)}=$$

$$\lim_{x\to1}\frac{x+13}{\left(x-1\right)\left(\sqrt{x}-1\right)}\overset{x\to1\Rightarrow x>0\Rightarrow x=\left(\sqrt{x}\right)^2}{=}\lim_{x\to1}\frac{x+13}{\left[\left(\sqrt{x}\right)^2-1\right]\left(\sqrt{x}-1\right)}=$$

$$=\lim_{x\to1}\frac{x+13}{\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)\left(\sqrt{x}-1\right)}=\lim_{x\to1}\frac{x+13}{\left(\sqrt{x}-1\right)^2\left(\sqrt{x}+1\right)}=$$

$$=\lim_{x\to1}\frac{1}{\left(\sqrt{x}-1\right)^2}\frac{x+13}{\sqrt{x}+1}=\lim_{x\to1}\frac{1}{\left(\sqrt{x}-1\right)^2}\lim_{x\to1}\frac{x+13}{\sqrt{x}+1}=\left(+\infty\right)7=+\infty$$

7.

$$\boxed{\mathrm{N}\alpha\ \beta\rho\varepsilon\theta\varepsilon\acute{\iota}\ \tau o\ \acute{o}\rho\iota o\ \lim_{x\to0}\frac{x+13}{2\left|x\right|+\left|x+3\right|-3}}$$

$\Theta\varepsilon\omega\rho\acute{\omega}\ \tau\eta\nu\ \sigma\upsilon\nu\acute{\alpha}\rho\tau\eta\sigma\eta: f\left(x\right)=\dfrac{x+13}{2\left|x\right|+\left|x+3\right|-3}$

$\mathrm{A}\nu\ x\in\left(-1,0\right)\ \theta\alpha\ \acute{\varepsilon}\chi\omega:$

$$\begin{cases} x<0 \\ x>-1 \end{cases} \Rightarrow \begin{cases} x<0 \\ x+3>2>0 \end{cases} \Rightarrow \begin{cases} x<0 \\ x+3>0 \end{cases} \Rightarrow \begin{cases} |x|=-x \\ |x+3|=x+3 \end{cases}$$

$$f(x)=\frac{x+13}{2|x|+|x+3|-3} \overset{\substack{|x|=-x \\ |x+3|=x+3}}{=} \frac{x+13}{2(-x)+x+3-3}=\frac{x+13}{-2x+x}=-\frac{1}{x}(x+13)$$

$$\lim_{x\to 0^-} f(x)=-\lim_{x\to 0^-}\frac{1}{x}\lim_{x\to 0^-}(x+13)=-(-\infty)13=+\infty$$

$\text{Αν } x\in(0,+\infty)\ \theta\alpha\ \acute{\varepsilon}\chi\omega:$

$$x>0 \Rightarrow \begin{cases} x>0 \\ x+3>3>0 \end{cases} \Rightarrow \begin{cases} x>0 \\ x+3>0 \end{cases} \Rightarrow \begin{cases} |x|=x \\ |x+3|=x+3 \end{cases}$$

$$f(x)=\frac{x+13}{2|x|+|x+3|-3} \overset{\substack{|x|=x \\ |x+3|=x+3}}{=} \frac{x+13}{2x+x+3-3}=\frac{x+13}{3x}=\frac{1}{3}\frac{1}{x}(x+13)$$

$$\lim_{x\to 0^+} f(x)=\frac{1}{3}\lim_{x\to 0^+}\frac{1}{x}\lim_{x\to 0^+}(x+13)=\frac{1}{3}(+\infty)13=+\infty$$

$\text{Επειδή } \lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=+\infty$ *υπάρχει το όριο της* f *στο σημείο* $x_0=0$

και ισχύει:

$$\lim_{x\to 0} f(x)=\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} f(x)=+\infty$$

8.

Να βρεθεί το όριο $\lim_{x\to 0}\frac{x+1}{\eta\mu x-x}$

Θεωρώ την συνάρτηση $f(x)=\frac{x+1}{\eta\mu x-x}, x\neq 0$

Αν $x\neq 0$ *θα έχω:* $|\eta\mu x|<|x|$

Αν $x>0$:

$$|\eta\mu x|<|x| \overset{|x|=x}{\Longleftrightarrow} |\eta\mu x|<x \Leftrightarrow -x<\eta\mu x<x \Rightarrow \eta\mu x-x<0$$

$$\acute{E}\chi\omega: \begin{cases} (\mathrm{I})\ \eta\mu x-x<0, x\in(0,+\infty) \\ (\mathrm{II})\lim_{x\to 0}(\eta\mu x-x)=0 \end{cases}$$

$$\text{Οπότε}: \lim_{x\to 0^+}\frac{1}{\eta\mu x-x}=-\infty$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{\eta\mu x - x}(x+1) = \lim_{x\to 0^+} \frac{1}{\eta\mu x - x} \lim_{x\to 0^+}(x+1) = (-\infty)\cdot 1 = -\infty$$

$\text{Αν } x<0:$

$$|\eta\mu x| < |x| \overset{|x|=-x}{\Longleftrightarrow} |\eta\mu x| < -x \Leftrightarrow -(-x) < \eta\mu x < -x \Rightarrow \eta\mu x > x \Rightarrow \eta\mu x - x > 0$$

$$\text{Έχω}: \left\{ \begin{matrix} (\text{I})\, \eta\mu x - x > 0, x \in (-\infty, 0) \\ (\text{II}) \lim_{x\to 0}(\eta\mu x - x) = 0 \end{matrix} \right\}$$

$$\text{Οπότε}: \lim_{x\to 0^-} \frac{1}{\eta\mu x - x} = +\infty$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{\eta\mu x - x}(x+1) = \lim_{x\to 0^-} \frac{1}{\eta\mu x - x} \lim_{x\to 0^+}(x+1) = (+\infty)\cdot 1 = +\infty$$

$\text{Επειδή } \lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^+} f(x)$ *δεν υπάρχει το όριο της* f *στο σημειο* $x_0 = 0$

ΑΣΚΗΣΕΙΣ

1.

Να βρεθεί το όριο $\lim_{x\to 0} \frac{x+3}{\sigma\upsilon\nu x - 1}$

2.

Να βρεθεί το όριο $\lim_{x\to \frac{\pi}{2}} \frac{x+11}{1-\eta\mu x}$

3.

Να βρεθεί το όριο $\lim_{x\to 0} \frac{x+18}{x^8 - 3x^4}$

4.

Να βρεθεί το όριο (Αν υπάρχει) $\lim_{x\to 0} \frac{x+15}{x^9 - x^5}$

5.

Να βρεθεί το όριο $\lim_{x\to 2} \frac{x+13}{|x-2| + 8(x-2)^2}$

6.

Να βρεθεί το όριο $\lim_{x\to 4} \frac{x-8}{x\sqrt{x} - 2x - 4\sqrt{x} + 8}$

7.

Να βρεθεί το όριο $\lim_{x\to 0} \frac{x+6}{2|x| + |x+5| - 5}$

8.

Να βρεθεί το όριο $\lim_{x\to \frac{\pi}{2}} \frac{x^2}{\sigma\upsilon\nu x + \frac{\pi}{2} - x}$