ΟΡΙΑ ΣΤΟ ΑΠΕΙΡΟ ΟΤΑΝ $x \to x_0$ ΚΑΙ ΑΝΙΣΩΣΕΙΣ

$\text{Αν}: \left\{ \begin{array}{l} (\text{I})\ f(x) \geq g(x)\ \ \gamma\iota\alpha\ \kappa\acute{\alpha}\theta\varepsilon\ x \in (\alpha, x_0) \cup (x_0, \beta) \\ (\text{II}) \lim\limits_{x \to x_0} g(x) = +\infty \end{array} \right\}$ $\text{Τ}\acute{o}\tau\varepsilon: \lim\limits_{x \to x_0} f(x) = +\infty$
$\text{Αν}: \left\{ \begin{array}{l} (\text{I})\ f(x) \leq g(x)\ \gamma\iota\alpha\ \kappa\acute{\alpha}\theta\varepsilon\ x \in (\alpha, x_0) \cup (x_0, \beta) \\ (\text{II}) \lim\limits_{x \to x_0} g(x) = -\infty \end{array} \right\}$ $\text{Τ}\acute{o}\tau\varepsilon: \lim\limits_{x \to x_0} f(x) = -\infty$

ΠΡΟΣΟΧΗ!!!
$\text{Α}\nu \lim\limits_{x \to x_0} f(x) = +\infty\ \kappa\alpha\iota\ \lvert g(x) \rvert \leq \theta, \theta > 0\ \tau\acute{o}\tau\varepsilon \lim\limits_{x \to x_0} [f(x) + g(x)] = +\infty$ $\Pi\rho\acute{\alpha}\gamma\mu\alpha\tau\iota:$ $\lvert g(x) \rvert \leq \theta \Leftrightarrow -\theta \leq g(x) \leq \theta \Leftrightarrow -\theta + f(x) \leq g(x) + f(x) \leq f(x) + \theta \Rightarrow$ $g(x) + f(x) \geq -\theta + f(x)$ $\left\{ \begin{array}{l} (\text{I})\ f(x) + g(x) \geq -\theta + f(x) \\ (\text{II}) \lim\limits_{x \to x_0} [f(x) - \theta] = +\infty \end{array} \right\}$ $\text{Ο}\pi\acute{o}\tau\varepsilon: \lim\limits_{x \to x_0} [f(x) + g(x)] = +\infty$
$\text{Α}\nu \lim\limits_{x \to x_0} f(x) = -\infty\ \kappa\alpha\iota\ \lvert g(x) \rvert \leq \theta, \theta > 0\ \tau\acute{o}\tau\varepsilon \lim\limits_{x \to x_0} [f(x) + g(x)] = -\infty$ $\Pi\rho\acute{\alpha}\gamma\mu\alpha\tau\iota:$ $\lvert g(x) \rvert \leq \theta \Leftrightarrow -\theta \leq g(x) \leq \theta \Leftrightarrow -\theta + f(x) \leq g(x) + f(x) \leq f(x) + \theta \Rightarrow$ $g(x) + f(x) \geq -\theta + f(x)$ $\left\{ \begin{array}{l} (\text{I})\ f(x) + g(x) \geq \theta + f(x) \\ (\text{II}) \lim\limits_{x \to x_0} [f(x) + \theta] = -\infty \end{array} \right\}$ $\text{Ο}\pi\acute{o}\tau\varepsilon: \lim\limits_{x \to x_0} [f(x) + g(x)] = -\infty$
$\lvert \eta\mu x \rvert \leq \lvert x \rvert$

ΠΑΡΑΔΕΙΓΜΑΤΑ

1.

$$\text{N}\alpha \ \beta\rho\varepsilon\theta\varepsilon\acute{\iota} \ \tau o \ \acute{o}\rho\iota o \ \lim_{x\to 0}\left(\frac{1}{x^2}+\eta\mu\frac{1}{x}\right)$$

$\text{A}\nu \ x\neq 0 \ \theta\alpha \ \acute{\varepsilon}\chi\omega:$

$$-1\leq \eta\mu\frac{1}{x}\leq 1 \Rightarrow -1+\frac{1}{x^2}\leq \eta\mu\frac{1}{x}+\frac{1}{x^2}\leq 1+\frac{1}{x^2}\Rightarrow \eta\mu\frac{1}{x}+\frac{1}{x^2}\geq -1+\frac{1}{x^2}$$

$$\acute{E}\chi\omega:\left\{\begin{matrix}(\text{I})\,\eta\mu\frac{1}{x}+\frac{1}{x^2}\geq -1+\frac{1}{x^2},\, x\in(-\infty,0)\cup(0,+\infty)\\ (\text{II})\lim_{x\to 0}\left(-1+\frac{1}{x^2}\right)=+\infty\end{matrix}\right\}$$

$$\text{O}\pi\acute{o}\tau\varepsilon:\lim_{x\to 0}\left(\eta\mu\frac{1}{x}+\frac{1}{x^2}\right)=+\infty$$

2.

$$\text{N}\alpha \ \beta\rho\varepsilon\theta\varepsilon\acute{\iota} \ \tau o \ \acute{o}\rho\iota o \ \lim_{x\to 0}\left(-\frac{1}{x^{10}}+\sigma\upsilon\nu\frac{1}{x}\right)$$

$\text{A}\nu \ x\neq 0 \ \theta\alpha \ \acute{\varepsilon}\chi\omega:$

$$-1\leq \sigma\upsilon\nu\frac{1}{x}\leq 1\Rightarrow -1-\frac{1}{x^{10}}\leq \sigma\upsilon\nu\frac{1}{x}-\frac{1}{x^{10}}\leq 1-\frac{1}{x^{10}}\Rightarrow \sigma\upsilon\nu\frac{1}{x}-\frac{1}{x^{10}}\leq 1-\frac{1}{x^{10}}$$

$$\lim_{x\to 0}\left(-\frac{1}{x^{10}}+\sigma\upsilon\nu\frac{1}{x}\right)$$

$$\acute{E}\chi\omega:\left\{\begin{matrix}(\text{I})\,\sigma\upsilon\nu\frac{1}{x}-\frac{1}{x^{10}}\leq 1-\frac{1}{x^{10}},\, x\in(-\infty,0)\cup(0,+\infty)\\ (\text{II})\lim_{x\to 0}\left(1-\frac{1}{x^{10}}\right)=-\infty\end{matrix}\right\}$$

$$\text{O}\pi\acute{o}\tau\varepsilon:\lim_{x\to 0}\left(-\frac{1}{x^{10}}+\sigma\upsilon\nu\frac{1}{x}\right)=-\infty$$

3.

$$\text{N}\alpha \ \beta\rho\varepsilon\theta\varepsilon\acute{\iota} \ \tau o \ \acute{o}\rho\iota o \ \lim_{x\to 0}\left(\frac{1}{x^4}+\frac{\eta\mu\frac{1}{x}}{1+\left|\eta\mu\frac{1}{x}\right|}\right)$$

$A\nu\ x \neq 0\ \theta\alpha\ \acute{\varepsilon}\chi\omega:$

$$\left|\frac{\eta\mu\frac{1}{x}}{1+\left|\eta\mu\frac{1}{x}\right|}\right| = \frac{\left|\eta\mu\frac{1}{x}\right|}{\left|1+\left|\eta\mu\frac{1}{x}\right|\right|} \overset{\left|1+\left|\eta\mu\frac{1}{x}\right|\right|=1+\left|\eta\mu\frac{1}{x}\right|}{=} \frac{\left|\eta\mu\frac{1}{x}\right|}{1+\left|\eta\mu\frac{1}{x}\right|} < 1$$

$$\acute{E}\chi\omega: \left|\eta\mu\frac{1}{x}\right| \geq 0 \Rightarrow \left|\eta\mu\frac{1}{x}\right| + 1 \geq 1 > 0 \Rightarrow \left|\eta\mu\frac{1}{x}\right| + 1 > 0 \Rightarrow \left|\left|\eta\mu\frac{1}{x}\right| + 1\right| = \left|\eta\mu\frac{1}{x}\right| + 1$$

$$\acute{E}\chi\omega: 1 > 0 \Rightarrow \left|\eta\mu\frac{1}{x}\right| + 1 > \left|\eta\mu\frac{1}{x}\right| \Rightarrow \left|\eta\mu\frac{1}{x}\right| < \left|\eta\mu\frac{1}{x}\right| + 1 \overset{1+\left|\eta\mu\frac{1}{x}\right|>0}{\Longrightarrow} \frac{\left|\eta\mu\frac{1}{x}\right|}{1+\left|\eta\mu\frac{1}{x}\right|} < 1$$

$$O\pi\acute{o}\tau\varepsilon: \left|\frac{\eta\mu\frac{1}{x}}{1+\left|\eta\mu\frac{1}{x}\right|}\right| < 1 \Rightarrow -1 + \frac{1}{x^4} < \frac{1}{x^4} + \frac{\eta\mu\frac{1}{x}}{1+\left|\eta\mu\frac{1}{x}\right|} < \frac{1}{x^4} + 1 \Rightarrow$$

$$\frac{1}{x^4} + \frac{\eta\mu\frac{1}{x}}{1+\left|\eta\mu\frac{1}{x}\right|} > -1 + \frac{1}{x^4}$$

$$\acute{E}\chi\omega: \left\{ \begin{array}{l} (\mathrm{I})\ \frac{1}{x^4} + \frac{\eta\mu\frac{1}{x}}{1+\left|\eta\mu\frac{1}{x}\right|} > -1 + \frac{1}{x^4},\ x \in (-\infty, 0) \cup (0, \infty) \\ (\mathrm{II})\ \lim\limits_{x\to 0}\left(-1 + \frac{1}{x^4}\right) = +\infty \end{array} \right\}$$

$$O\pi\acute{o}\tau\varepsilon: \lim_{x\to 0}\left(\frac{1}{x^4} + \frac{\eta\mu\frac{1}{x}}{1+\left|\eta\mu\frac{1}{x}\right|}\right) = +\infty$$

4.

$N\alpha\ \beta\rho\varepsilon\theta\varepsilon\acute{\iota}\ \tau o\ \acute{o}\rho\iota o\ \lim\limits_{x\to 0}\left(-\frac{1}{x^6} + \frac{\eta\mu\left(2 - \left\lvert\eta\mu\frac{1}{x}\right\rvert\right)}{3 - \left\lvert\eta\mu\frac{1}{x}\right\rvert}\right)$

Αν $x \neq 0$ θα έχω:

$$\left|\frac{\eta\mu\left(2-\left|\eta\mu\frac{1}{x}\right|\right)}{3-\left|\eta\mu\frac{1}{x}\right|}\right| = \frac{\left|\eta\mu\left(2-\left|\eta\mu\frac{1}{x}\right|\right)\right|}{\left|3-\left|\eta\mu\frac{1}{x}\right|\right|} \overset{|\eta\mu x|\leq|x|}{\leq} \frac{\left|2-\left|\eta\mu\frac{1}{x}\right|\right|}{\left|3-\left|\eta\mu\frac{1}{x}\right|\right|} = \frac{2-\left|\eta\mu\frac{1}{x}\right|}{3-\left|\eta\mu\frac{1}{x}\right|} < 1$$

$$\left|\eta\mu\frac{1}{x}\right| \leq 1 < 2 \Rightarrow 2 > \left|\eta\mu\frac{1}{x}\right| \Rightarrow 2-\left|\eta\mu\frac{1}{x}\right| > 0 \Rightarrow \left|2-\left|\eta\mu\frac{1}{x}\right|\right| = 2-\left|\eta\mu\frac{1}{x}\right|$$

$$\left|\eta\mu\frac{1}{x}\right| \leq 1 < 3 \Rightarrow 3 > \left|\eta\mu\frac{1}{x}\right| \Rightarrow 3-\left|\eta\mu\frac{1}{x}\right| > 0 \Rightarrow \left|3-\left|\eta\mu\frac{1}{x}\right|\right| = 3-\left|\eta\mu\frac{1}{x}\right|$$

$$2 < 3 \Rightarrow 2-\left|\eta\mu\frac{1}{x}\right| < 3-\left|\eta\mu\frac{1}{x}\right| \overset{3-\left|\eta\mu\frac{1}{x}\right|>0}{\Longrightarrow} \frac{2-\left|\eta\mu\frac{1}{x}\right|}{3-\left|\eta\mu\frac{1}{x}\right|} < \frac{3-\left|\eta\mu\frac{1}{x}\right|}{3-\left|\eta\mu\frac{1}{x}\right|} \Rightarrow \frac{2-\left|\eta\mu\frac{1}{x}\right|}{3-\left|\eta\mu\frac{1}{x}\right|} < 1$$

$$\left|\frac{\eta\mu\left(2-\left|\eta\mu\frac{1}{x}\right|\right)}{3-\left|\eta\mu\frac{1}{x}\right|}\right| < 1 \Rightarrow -1 \leq \frac{\eta\mu\left(2-\left|\eta\mu\frac{1}{x}\right|\right)}{3-\left|\eta\mu\frac{1}{x}\right|} < 1 \Rightarrow -1-\frac{1}{x^6} \leq -\frac{1}{x^6} + \frac{\eta\mu\left(2-\left|\eta\mu\frac{1}{x}\right|\right)}{3-\left|\eta\mu\frac{1}{x}\right|} \leq 1-\frac{1}{x^6} \Rightarrow$$

$$-\frac{1}{x^6} + \frac{\eta\mu\left(2-\left|\eta\mu\frac{1}{x}\right|\right)}{3-\left|\eta\mu\frac{1}{x}\right|} \leq 1-\frac{1}{x^6}$$

$$\text{Έχω}: \left\{ \begin{array}{l} (\mathrm{I})\, -\frac{1}{x^6} + \frac{\eta\mu\left(2-\left|\eta\mu\frac{1}{x}\right|\right)}{3-\left|\eta\mu\frac{1}{x}\right|} \leq 1-\frac{1}{x^6},\, x \in \left(-\frac{\pi}{2}, 0\right) \cup (0, +\infty) \\ (\mathrm{II}) \lim_{x\to 0}\left(1-\frac{1}{x^6}\right) = -\infty \end{array} \right\}$$

$$\text{Οπότε}: \lim_{x\to 0}\left(-\frac{1}{x^6} + \frac{\eta\mu\left(2-\left|\eta\mu\frac{1}{x}\right|\right)}{3-\left|\eta\mu\frac{1}{x}\right|}\right) = -\infty$$

<u>ΑΣΚΗΣΕΙΣ</u>

1.

Να βρεθεί το όριο $\lim_{x\to 0}\left(-\frac{1}{x^2} + \eta\mu\frac{1}{x}\right)$

2.

Να βρεθεί το όριο $\lim_{x\to 0}\left(\frac{1}{x^{16}} + \sigma\upsilon\nu\frac{1}{x}\right)$

3.

Να βρεθεί το όριο $\lim_{x\to 0}\left(-\frac{1}{x^8} + \frac{\eta\mu(1/x)}{1+\left|\eta\mu(1/x)\right|}\right)$