ΟΡΙΑ ΜΕ ΡΙΖΙΚΑ ΔΕΥΤΕΡΗΣ ΚΑΙ ΤΡΙΤΗΣ ΤΑΞΗΣ

Πως θα βρω το $\lim_{x\to x_0}\dfrac{\alpha\sqrt[\kappa]{f(x)}+\beta\sqrt[\lambda]{g(x)}-L}{x-x_0}$, όταν:

$\lim_{x\to x_0}\sqrt[\kappa]{f(x)}=L_1\ \lim_{x\to x_0}\sqrt[\lambda]{g(x)}=L_2,\ L=\alpha L_1+\beta L_2,\ \kappa,\lambda\in\{2,3\}$

↓

$$\frac{\alpha\sqrt[\kappa]{f(x)}+\beta\sqrt[\lambda]{g(x)}-L}{x-x_0}\overset{L=\alpha L_1+\beta L_2}{=}\frac{\alpha\sqrt[\kappa]{f(x)}+\beta\sqrt[\lambda]{g(x)}-\alpha L_1-\beta L_2}{x-x_0}=$$

$$\alpha\frac{\sqrt[\kappa]{f(x)}-L_1}{x-x_0}+\beta\frac{\sqrt[\lambda]{g(x)}-L_2}{x-x_0}$$

Οπότε: $\lim_{x\to x_0}\dfrac{\alpha\sqrt[\kappa]{f(x)}+\beta\sqrt[\lambda]{g(x)}-L}{x-x_0}=\alpha\lim_{x\to x_0}\dfrac{\sqrt[\kappa]{f(x)}-L_1}{x-x_0}+\beta\lim_{x\to x_0}\dfrac{\sqrt[\lambda]{g(x)}-L_2}{x-x_0}$

ΠΑΡΑΔΕΙΓΜΑΤΑ

1.

Δίνεται η συνάρτηση $f(x)=\dfrac{\sqrt{x}+2\sqrt[3]{x}-3}{x-1},\ x\in(0,1)\cup(1,+\infty)$

Να βρεθεί το όριο της f στο σημείο $x_0=1$

$$f(x)=\frac{\sqrt{x}+2\sqrt[3]{x}-3}{x-1}\overset{\substack{-3=-1-2\\ \lim_{x\to1}\sqrt{x}=\lim_{x\to1}\sqrt[3]{x}=1}}{=}\frac{\sqrt{x}-1+2\sqrt[3]{x}-2}{x-1}=\frac{\sqrt{x}-1+2\left(\sqrt[3]{x}-1\right)}{x-1}=$$

$$=\frac{\sqrt{x}-1}{x-1}+2\frac{\sqrt[3]{x}-1}{x-1}$$

$$\frac{\sqrt{x}-1}{x-1}\overset{\substack{\text{Πολλαπλασιάζω αριθμητή}\\ \text{και παρονομαστή με }\sqrt{x}-1}}{=}\frac{\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}{(x-1)\left(\sqrt{x}+1\right)}\overset{(\alpha-\beta)(\alpha+\beta)=\alpha^2-\beta^2}{=}\frac{\left(\sqrt{x}\right)^2-1}{(x-1)\left(\sqrt{x}+1\right)}$$

$$\overset{\left(\sqrt{\alpha}\right)^2=\alpha,\alpha\ge0}{=}\frac{\cancel{x-1}}{\cancel{(x-1)}\left(\sqrt{x}+1\right)}=\frac{1}{\sqrt{x}+1}$$

$$\lim_{x\to1}\frac{\sqrt{x}-1}{x-1}=\lim_{x\to1}\frac{1}{\sqrt{x}+1}=\frac{1}{\sqrt{1}+1}=\frac{1}{2}$$

$$\frac{\sqrt[3]{x}-1}{x-1} \overset{\substack{\text{Πολλαπλασιάζω αριθμητή} \\ \text{και παρονομαστή με } \left(\sqrt[3]{x}\right)^2+\sqrt[3]{x}+1}}{=} \frac{\left(\sqrt[3]{x}-1\right)\left[\left(\sqrt[3]{x}\right)^2+\sqrt[3]{x}+1\right]}{(x-1)\left[\left(\sqrt[3]{x}\right)^2+\sqrt[3]{x}+1\right]} \overset{(\alpha-\beta)\left(\alpha^2+\alpha\beta+\beta^2\right)=\alpha^3-\beta^3}{=}$$

$$\frac{\left(\sqrt[3]{x}\right)^3-1}{(x-1)\left[\left(\sqrt[3]{x}\right)^2+\sqrt[3]{x}+1\right]} \overset{\left(\sqrt[3]{\alpha}\right)^3=\alpha,\alpha\geq 0}{=} \frac{\cancel{x-1}}{\cancel{(x-1)}\left[\left(\sqrt[3]{x}\right)^2+\sqrt[3]{x}+1\right]} = \frac{1}{\left(\sqrt[3]{x}\right)^2+\sqrt[3]{x}+1}$$

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1} = \lim_{x\to 1}\frac{1}{\left(\sqrt[3]{x}\right)^2+\sqrt[3]{x}+1} = \frac{1}{\left(\sqrt[3]{1}\right)^2+\sqrt[3]{1}+1} = \frac{1}{3}$$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}\left(\frac{\sqrt{x}-1}{x-1}+2\frac{\sqrt[3]{x}-1}{x-1}\right) = \lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}+2\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1} = \frac{1}{2}+\frac{2}{3} =$$

$$= \frac{3}{6}+\frac{4}{6} = \frac{7}{6}$$

2.

Δίνεται η συνάρτηση $f(x) = \dfrac{\sqrt{x^2+16}+2\sqrt[3]{x+5}-9}{x-3}, x\in(0,3)\cup(3,+\infty)$
Να βρεθεί το όριο της f *στο σημείο* $x_0 = 3$

$$f(x) = \frac{\sqrt{x^2+16}+2\sqrt[3]{x+5}-9}{x-3} \overset{\substack{-9=-5-4 \\ \lim_{x\to 3}\sqrt{x^2+16}=5 \\ \lim_{x\to 3}\sqrt[3]{x+5}=2}}{=} \frac{\sqrt{x^2+16}-5+2\sqrt[3]{x+5}-4}{x-3} =$$

$$\frac{\sqrt{x^2+16}-5}{x-3}+\frac{2\sqrt[3]{x+5}-4}{x-3} = \frac{\sqrt{x^2+16}-5}{x-3}+2\frac{\sqrt[3]{x+5}-2}{x-3}$$

$$\frac{\sqrt{x^2+16}-5}{x-3} \overset{\substack{\text{Πολλαπλασιάζω αριθμητή} \\ \text{και παρονομαστή με } \sqrt{x^2+16}-5}}{=} \frac{\left(\sqrt{x^2+16}-5\right)\left(\sqrt{x^2+16}+5\right)}{(x-3)\left(\sqrt{x^2+16}+5\right)} \overset{(a-\beta)(a+\beta)=\alpha^2-\beta^2}{=}$$

$$\frac{\left(\sqrt{x^2+16}\right)^2-25}{(x-3)\left(\sqrt{x^2+16}+5\right)} = \frac{x^2-9}{(x-3)\left(\sqrt{x^2+16}+5\right)} \overset{x^2-9=(x-3)(x+3)}{=} \frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}\left(\sqrt{x^2+16}+5\right)}$$

$$= \frac{x+3}{\sqrt{x^2+16}+5}$$

$$\lim_{x\to 3}\frac{\sqrt{x^2+16}-5}{x-3}=\lim_{x\to 3}\frac{x+3}{\sqrt{x^2+16}+5}=\frac{3+3}{\sqrt{3^2+16}+5}=\frac{9}{10}$$

$$\frac{\sqrt[3]{x+5}-2}{x-3}\overset{\substack{\text{Πολλαπλασιάζω αριθμητή}\\ \text{και παρονομαστή με}\\ \left(\sqrt[3]{x+5}\right)^2+2\sqrt[3]{x+5}+2^2}}{=}\frac{\left(\sqrt[3]{x+5}-2\right)\left[\left(\sqrt[3]{x+5}\right)^2+2\sqrt[3]{x+5}+2^2\right]}{(x-3)\left[\left(\sqrt[3]{x+5}\right)^2+2\sqrt[3]{x+5}+2^2\right]}\overset{(\alpha-\beta)\left(\alpha^2+\alpha\beta+\beta^2\right)=\alpha^3-\beta^3}{=}$$

$$\frac{\left(\sqrt[3]{x+5}\right)^3-2^3}{(x-3)\left[\left(\sqrt[3]{x+5}\right)^2+2\sqrt[3]{x+5}+4\right]}=\frac{\cancel{x-3}}{\cancel{(x-3)}\left[\left(\sqrt[3]{x+5}\right)^2+2\sqrt[3]{x+5}+4\right]}=$$

$$=\frac{1}{\left(\sqrt[3]{x+5}\right)^2+2\sqrt[3]{x+5}+4}$$

$$\lim_{x\to 3}\frac{\sqrt[3]{x+5}-2}{x-3}=\lim_{x\to 3}\frac{1}{\left(\sqrt[3]{x+5}\right)^2+2\sqrt[3]{x+5}+4}=\frac{1}{\left(\sqrt[3]{3+5}\right)^2+2\sqrt[3]{3+5}+4}=\frac{1}{12}$$

$$\lim_{x\to 3}f(x)=\lim_{x\to 3}\left(\frac{\sqrt{x^2+16}-5}{x-3}+2\frac{\sqrt[3]{x+5}-2}{x-3}\right)=\lim_{x\to 3}\frac{\sqrt{x^2+16}-5}{x-3}+2\lim_{x\to 3}\frac{\sqrt[3]{x+5}-2}{x-3}=$$

$$=\frac{9}{10}+2\bullet\frac{1}{12}=\frac{9}{10}+\frac{1}{6}=\frac{27}{30}+\frac{5}{30}=\frac{32:2}{30:2}=\frac{16}{15}$$

3.

Δίνεται η συνάρτηση $f(x)=\frac{\sqrt{x}-\sqrt[3]{x}}{x-1},\ x\in(0,1)\cup(1,+\infty)$
Να βρεθεί το όριο της f στο σημείο $x_0=1$

$$f(x)=\frac{\sqrt{x}-\sqrt[3]{x}}{x-1}\overset{\substack{\text{Προσθαφαιρώ το 1}\\ \lim_{x\to1}\sqrt{x}=\lim_{x\to1}\sqrt[3]{x}=1}}{=}\frac{\sqrt{x}-1+\underbrace{1-\sqrt[3]{x}}_{-\left(\sqrt[3]{x}-1\right)}}{x-1}=\frac{\sqrt{x}-1-\left(\sqrt[3]{x}-1\right)}{x-1}=\frac{\sqrt{x}-1}{x-1}-\frac{\sqrt[3]{x}-1}{x-1}$$

$$\frac{\sqrt{x}-1}{x-1}\overset{\substack{\text{Πολλαπλασιάζω αριθμητή}\\ \text{και παρονομαστή με }\sqrt{x}-1}}{=}\frac{\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}{(x-1)\left(\sqrt{x}+1\right)}\overset{(\alpha-\beta)(\alpha+\beta)=\alpha^2-\beta^2}{=}\frac{\left(\sqrt{x}\right)^2-1}{(x-1)\left(\sqrt{x}+1\right)}$$

$$\overset{\left(\sqrt{\alpha}\right)^2=\alpha,\alpha\ge 0}{=}\frac{\cancel{x-1}}{\cancel{(x-1)}\left(\sqrt{x}+1\right)}=\frac{1}{\sqrt{x}+1}$$

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}=\lim_{x\to 1}\frac{1}{\sqrt{x}+1}=\frac{1}{\sqrt{1}+1}=\frac{1}{2}$$

$$\frac{\sqrt[3]{x}-1}{x-1}\overset{\substack{\text{Πολλαπλασιάζω αριθμητή}\\ \text{και παρονομαστή με}\left(\sqrt[3]{x}\right)^2+\sqrt[3]{x}+1}}{=}\frac{\left(\sqrt[3]{x}-1\right)\left[\left(\sqrt[3]{x}\right)^2+\sqrt[3]{x}+1\right]}{(x-1)\left[\left(\sqrt[3]{x}\right)^2+\sqrt[3]{x}+1\right]}\overset{(\alpha-\beta)\left(\alpha^2+\alpha\beta+\beta^2\right)=\alpha^3-\beta^3}{=}$$

$$\frac{\left(\sqrt[3]{x}\right)^3-1}{(x-1)\left[\left(\sqrt[3]{x}\right)^2+\sqrt[3]{x}+1\right]}\overset{\left(\sqrt[3]{\alpha}\right)^3=\alpha,\alpha\geq 0}{=}\frac{\cancel{x-1}}{\cancel{(x-1)}\left[\left(\sqrt[3]{x}\right)^2+\sqrt[3]{x}+1\right]}=\frac{1}{\left(\sqrt[3]{x}\right)^2+\sqrt[3]{x}+1}$$

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1}=\lim_{x\to 1}\frac{1}{\left(\sqrt[3]{x}\right)^2+\sqrt[3]{x}+1}=\frac{1}{\left(\sqrt[3]{1}\right)^2+\sqrt[3]{1}+1}=\frac{1}{3}$$

$$\lim_{x\to 1}f(x)=\lim_{x\to 1}\left(\frac{\sqrt{x}-1}{x-1}-\frac{\sqrt[3]{x}-1}{x-1}\right)=\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}-\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1}=\frac{1}{2}-\frac{1}{3}=\frac{3}{6}-\frac{2}{6}=\frac{1}{6}$$

4.

Δίνεται η συνάρτηση $f(x)=\dfrac{\sqrt{x+4}-\sqrt[3]{x+22}}{x-5}, x\in(0,5)\cup(5,+\infty)$

Να βρεθεί το όριο της f στο σημείο $x_0=5$

$$f(x)=\frac{\sqrt{x+4}-\sqrt[3]{x+22}}{x-5}\overset{\substack{\text{Προσθαφαιρώ το 3}\\ \lim_{x\to 5}\sqrt{x+4}=\lim_{x\to 5}\sqrt[3]{x+22}=3}}{=}\frac{\sqrt{x+4}-3+\underbrace{3-\sqrt[3]{x+22}}_{-\left(\sqrt[3]{x+22}-3\right)}}{x-5}=$$

$$=\frac{\sqrt{x+4}-3-\left(\sqrt[3]{x+22}-3\right)}{x-5}=\frac{\sqrt{x+4}-3}{x-5}-\frac{\sqrt[3]{x+22}-3}{x-5}$$

$$\frac{\sqrt{x+4}-3}{x-5}\overset{\substack{\text{Πολλαπλασιάζω αριθμητή}\\ \text{και παρονομαστή με }\sqrt{x+4}+3}}{=}\frac{\left(\sqrt{x+4}-3\right)\left(\sqrt{x+4}+3\right)}{(x-5)\left(\sqrt{x+4}+3\right)}\overset{(\alpha-\beta)(\alpha+\beta)=\alpha^2-\beta^2}{=}$$

$$\frac{\left(\sqrt{x+4}\right)^2-9}{(x-5)\left(\sqrt{x+4}+3\right)}=\frac{\cancel{x-5}}{\cancel{(x-5)}\left(\sqrt{x+4}+3\right)}=\frac{1}{\sqrt{x+4}+3}$$

$$\lim_{x\to 5}\frac{\sqrt{x+4}-3}{x-5}=\lim_{x\to 5}\frac{1}{\sqrt{x+4}+3}=\frac{1}{\sqrt{5+4}+3}=\frac{1}{6}$$

$$\frac{\sqrt[3]{x+22}-3}{x-5} \overset{\substack{\text{Πολλαπλασιάζω αριθμητή} \\ \text{και παρονομαστή με} \\ \left(\sqrt[3]{x+22}\right)^2+\sqrt[3]{x+22}\cdot 3+3^2}}{=} \frac{\left(\sqrt[3]{x+22}-3\right)\left[\left(\sqrt[3]{x+22}\right)^2+\sqrt[3]{x+22}\cdot 3+3^2\right]}{(x-5)\left[\left(\sqrt[3]{x+22}\right)^2+\sqrt[3]{x+22}\cdot 3+3^2\right]}$$

$$\overset{(\alpha-\beta)\left(\alpha^2+\alpha\beta+\beta^2\right)=\alpha^3-\beta^3}{=} \frac{\left(\sqrt[3]{x+22}\right)^3-27}{(x-5)\left[\left(\sqrt[3]{x+22}\right)^2+3\sqrt[3]{x+22}+9\right]}=$$

$$\frac{\cancel{x-5}}{\cancel{(x-5)}\left[\left(\sqrt[3]{x+22}\right)^2+3\sqrt[3]{x+22}+9\right]}=\frac{1}{\left(\sqrt[3]{x+22}\right)^2+3\sqrt[3]{x+22}+9}$$

$$\lim_{x\to 5}\frac{\sqrt[3]{x+22}-3}{x-5}=\lim_{x\to 5}\frac{1}{\left(\sqrt[3]{x+22}\right)^2+3\sqrt[3]{x+22}+9}=\frac{1}{\left(\sqrt[3]{5+22}\right)^2+3\sqrt[3]{5+22}+9}=\frac{1}{27}$$

$$\lim_{x\to 5} f(x)=\lim_{x\to 5}\left(\frac{\sqrt{x+4}-3}{x-5}-\frac{\sqrt[3]{x+22}-3}{x-5}\right)=\lim_{x\to 5}\frac{\sqrt{x+4}-3}{x-5}-\lim_{x\to 5}\frac{\sqrt[3]{x+22}-3}{x-5}=$$

$$=\frac{1}{6}-\frac{1}{27}=\frac{9}{54}-\frac{2}{54}=\frac{7}{54}$$

5.

Δίνεται η συνάρτηση $f(x)=\dfrac{\sqrt{\eta\mu x+1}-\sqrt[3]{\eta\mu x+1}}{\eta\mu x}, x\in\left(-\dfrac{\pi}{2},0\right)\cup\left(0,\dfrac{\pi}{2}\right)$

Να βρεθεί το όριο της f στο σημείο $x_0=0$

$$f(x)=\frac{\sqrt{\eta\mu x+1}-\sqrt[3]{\eta\mu x+1}}{\eta\mu x} \overset{\substack{\text{Προσθαφαιρώ το 1} \\ \lim_{x\to 0}\sqrt{\eta\mu x+1}=\lim_{x\to 0}\sqrt[3]{\eta\mu x+1}=1}}{=} \frac{\sqrt{\eta\mu x+1}-1+\underbrace{1-\sqrt[3]{\eta\mu x+1}}_{-\left(\sqrt[3]{\eta\mu x+1}-1\right)}}{\eta\mu x}=$$

$$\frac{\sqrt{\eta\mu x+1}-1-\left(\sqrt[3]{\eta\mu x+1}-1\right)}{\eta\mu x}=\frac{\sqrt{\eta\mu x+1}-1}{\eta\mu x}-\frac{\sqrt[3]{\eta\mu x+1}-1}{\eta\mu x}$$

$$\frac{\sqrt{\eta\mu x+1}-1}{\eta\mu x}=\frac{\left(\sqrt{\eta\mu x+1}-1\right)\left(\sqrt{\eta\mu x+1}+1\right)}{\eta\mu x\left(\sqrt{\eta\mu x+1}+1\right)}=\frac{\left(\sqrt{\eta\mu x+1}\right)^2-1}{\eta\mu x\left(\sqrt{\eta\mu x+1}+1\right)}=$$

$$=\frac{\cancel{\eta\mu x}}{\cancel{\eta\mu x}\left(\sqrt{\eta\mu x+1}+1\right)}=\frac{1}{\sqrt{\eta\mu x+1}+1}$$

$$\lim_{x\to0}\frac{\sqrt{\eta\mu x+1}-1}{\eta\mu x}=\lim_{x\to0}\frac{1}{\sqrt{\eta\mu x+1}+1}=\frac{1}{\sqrt{\eta\mu 0+1}+1}=\frac{1}{2}$$

$$\frac{\sqrt[3]{\eta\mu x+1}-1}{\eta\mu x}=\frac{\left(\sqrt[3]{\eta\mu x+1}-1\right)\left[\left(\sqrt[3]{\eta\mu x+1}\right)^2+\sqrt[3]{\eta\mu x+1}+1\right]}{\eta\mu x\left[\left(\sqrt[3]{\eta\mu x+1}\right)^2+\sqrt[3]{\eta\mu x+1}+1\right]}=$$

$$\frac{\left(\sqrt[3]{\eta\mu x+1}\right)^3-1}{\eta\mu x\left[\left(\sqrt[3]{\eta\mu x+1}\right)^2+\sqrt[3]{\eta\mu x+1}+1\right]}=\frac{\cancel{\eta\mu x}}{\cancel{\eta\mu x}\left[\left(\sqrt[3]{\eta\mu x+1}\right)^2+\sqrt[3]{\eta\mu x+1}+1\right]}=$$

$$\frac{1}{\left(\sqrt[3]{\eta\mu x+1}\right)^2+\sqrt[3]{\eta\mu x+1}+1}$$

$$\lim_{x\to0}\frac{\sqrt[3]{\eta\mu x+1}-1}{\eta\mu x}=\lim_{x\to0}\frac{1}{\left(\sqrt[3]{\eta\mu x+1}\right)^2+\sqrt[3]{\eta\mu x+1}+1}=\frac{1}{\left(\sqrt[3]{\eta\mu 0+1}\right)^2+\sqrt[3]{\eta\mu 0+1}+1}=\frac{1}{3}$$

$$\lim_{x\to0}f(x)=\lim_{x\to0}\frac{\sqrt{\eta\mu x+1}-1}{\eta\mu x}-\lim_{x\to0}\frac{\sqrt[3]{\eta\mu x+1}-1}{\eta\mu x}=\frac{1}{2}-\frac{1}{3}=\frac{3}{6}-\frac{2}{6}=\frac{1}{6}$$

ΑΣΚΗΣΕΙΣ

1.

$N\alpha\ \beta\rho\varepsilon\theta\varepsilon\acute{\iota}\ \tau o\ \acute{o}\rho\iota o\ \lim_{x\to1}\frac{\sqrt{x}+5\sqrt[3]{x}-6}{x-1}$

2.

$N\alpha\ \beta\rho\varepsilon\theta\varepsilon\acute{\iota}\ \tau o\ \acute{o}\rho\iota o\ \lim_{x\to2}\frac{\sqrt{x^2+5}+\sqrt[3]{x+6}-5}{x-2}$

3.

$N\alpha\ \beta\rho\varepsilon\theta\varepsilon\acute{\iota}\ \tau o\ \acute{o}\rho\iota o\ \lim_{x\to4}\frac{\sqrt{x}-\sqrt[3]{x+4}}{x-4}$

4.

$N\alpha\ \beta\rho\varepsilon\theta\varepsilon\acute{\iota}\ \tau o\ \acute{o}\rho\iota o\ \lim_{x\to6}\frac{\sqrt{x+3}-\sqrt[3]{x+21}}{x-6}$

5.

$N\alpha\ \beta\rho\varepsilon\theta\varepsilon\acute{\iota}\ \tau o\ \acute{o}\rho\iota o\ \lim_{x\to0}\frac{\sqrt{1-\eta\mu x}-\sqrt[3]{1-\eta\mu x}}{\eta\mu x}$