$\Pi \omega\varsigma\ \theta\alpha\ \beta\rho\omega\ \tau o \lim_{x\to x_0} f(x)$ *όταν δίνεται μια σχέση της μορφής*
$\lim_{x\to x_0} g(f(x)) = L$ *όπου* $g(f(x))$: *Παράσταση που εμφανίζεται η μεταβλητή x και η συνάρτηση* $f(x)$

↓

Θέτω: $h(x) = g(f(x))\ (1)$ *τότε θα έχω* $\lim_{x\to x_0} h(x) = L$

↓

Λύνω την σχέση $h(x) = g(f(x))$ *ως προς* $f(x)$. *Τότε θα έχω*:
$f(x) = w(h(x))$

↓

Στην σχέση $f(x) = w(h(x))$ *παίρνω το όριο της* f *στο* x_0 *λαμβάνοντας υπόψη οτι* $\lim_{x\to x_0} h(x) = L$

ΠΑΡΑΔΕΙΓΜΑΤΑ

1.

Δίνεται η συνάρτηση $f: [-1,0)\cup(0,+\infty)$. *Να βρεθεί το* $\lim_{x\to 0} f(x)$ *όταν*

$$\lim_{x\to 0} \frac{x\left(\sqrt{x+1}-1\right) f(x)}{\sqrt{x^2+1}-1} = 2$$

Θεωρώ την συνάρτηση $g: [-1,0)\cup(0,+\infty)$ *με τύπο* $g(x) = \dfrac{\left(\sqrt{x+1}-1\right) f(x)}{\sqrt{x^2+1}-1}$

Τότε θα έχω $\lim_{x\to 0} g(x) = 2$

$$\left\{\begin{array}{l} g(x) = \dfrac{x\left(\sqrt{x+1}-1\right) f(x)}{\sqrt{x^2+1}-1} \\ x\in[-1,0)\cup(0,+\infty) \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} \left(\sqrt{x^2+1}-1\right) g(x) = x\left(\sqrt{x+1}-1\right) f(x) \\ x\in[-1,0)\cup(0,+\infty) \end{array}\right\}$$

Θεωρώ την εξίσωση: $\left\{\begin{array}{l} x\left(\sqrt{x+1}-1\right) = 0 \\ x\in[-1,0)\cup(0,+\infty) \end{array}\right\} \Leftrightarrow$

$$\left\{\begin{pmatrix}\sqrt{x+1}-1=0\\ x\in[-1,0)\cup(0,+\infty)\end{pmatrix} \acute{\eta} \begin{pmatrix}x=0\\ x\in[-1,0)\cup(0,+\infty)\\ (\textit{Άτοπο})\end{pmatrix}\right\} \Leftrightarrow \left\{\begin{matrix}\sqrt{x+1}=1\\ x\in[-1,0)\cup(0,+\infty)\end{matrix}\right\} \Leftrightarrow$$

$$\left\{\begin{matrix}\left(\sqrt{x+1}\right)^2=1^2\\ x\in[-1,0)\cup(0,+\infty)\end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix}x+1=1\\ x\in[-1,0)\cup(0,+\infty)\end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix}x=0\\ x\in[-1,0)\cup(0,+\infty)\\ (\textit{Άτοπο})\end{matrix}\right\}$$

$\Sigma\upsilon\nu\varepsilon\pi\acute{\omega}\varsigma\ x\left(\sqrt{x+1}-1\right)\neq 0\ \gamma\iota\alpha\ \kappa\acute{\alpha}\theta\varepsilon\ x\in[-1,0)\cup(0,+\infty)$

$$\left\{\begin{matrix}x\left(\sqrt{x+1}-1\right)f(x)=\left(\sqrt{x^2+1}-1\right)g(x)\\ x\in[-1,0)\cup(0,+\infty)\end{matrix}\right\} \overset{x\left(\sqrt{x+1}-1\right)\neq 0\ \gamma\iota\alpha\ \kappa\acute{\alpha}\theta\varepsilon\ x\in[-1,0)\cup(0,+\infty)}{\Leftrightarrow}$$

$$\left\{\begin{matrix}f(x)=\dfrac{\sqrt{x^2+1}-1}{x\left(\sqrt{x+1}-1\right)}g(x)\\ x\in[-1,0)\cup(0,+\infty)\end{matrix}\right\}$$

$$f(x)=\frac{\left(\sqrt{x^2+1}-1\right)\left(\sqrt{x^2+1}+1\right)\left(\sqrt{x+1}+1\right)}{x\left(\sqrt{x+1}-1\right)\left(\sqrt{x+1}+1\right)\left(\sqrt{x^2+1}+1\right)}g(x)=$$

$$=\frac{\left[\left(\sqrt{x^2+1}\right)^2-1\right]}{x\left[\left(\sqrt{x+1}\right)^2-1^2\right]\left(\sqrt{x^2+1}+1\right)}g(x)$$

$$=\frac{x^2+1-1}{x(x+1-1)\left(\sqrt{x^2+1}+1\right)}g(x)=\frac{x^2}{x^2\left(\sqrt{x^2+1}+1\right)}g(x)=\frac{g(x)}{\sqrt{x^2+1}+1}$$

$$\lim_{x\to 0}f(x)=\lim_{x\to 0}\frac{g(x)}{\sqrt{x^2+1}+1}=\frac{\lim\limits_{x\to 0}g(x)}{\lim\limits_{x\to 0}\left(\sqrt{x^2+1}+1\right)}=\frac{2}{\sqrt{0^2+1}+1}=\frac{2}{2}=1$$

2.

> $\Delta\acute{\iota}\nu\varepsilon\tau\alpha\iota\ \eta\ \sigma\upsilon\nu\acute{\alpha}\rho\tau\eta\sigma\eta\ f:[-6,3)\cup(3,+\infty)\to\mathbb{R}\ \mu\varepsilon\ \lim\limits_{x\to 3}\dfrac{f(x)-x}{\sqrt{x+6}-3}=1$
>
> $\mathrm{N}\alpha\ \beta\rho\varepsilon\theta o\acute{\upsilon}\nu\ \tau\alpha\ \acute{o}\rho\iota\alpha:$
>
> $(\mathrm{I})\lim\limits_{x\to 3}f(x)\quad (\mathrm{II})\lim\limits_{x\to 3}\dfrac{f(x)-3}{x^2-9}\ (\mathrm{III})\ \lim\limits_{x\to 3}\dfrac{f^2(x)-9}{f^3(x)-3f^2(x)+2f(x)-6}$

(I) Θεωρώ την συνάρτηση $g:[-6,3)\cup(3,+\infty)$ με τύπο $g(x)=\dfrac{f(x)-x}{\sqrt{x+6}-3}$

Τότε θα έχω $\lim\limits_{x\to 3} g(x)=1$

$$\left\{\begin{array}{l} g(x)=\dfrac{f(x)-x}{\sqrt{x+6}-3} \\ x\in[-6,3)\cup(3,+\infty)\end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} f(x)-x=g(x)\left(\sqrt{x+6}-3\right) \\ x\in[-6,3)\cup(3,+\infty)\end{array}\right\} \Leftrightarrow$$

$$\left\{\begin{array}{l} f(x)=g(x)\left(\sqrt{x+6}-3\right)+x \\ x\in[-6,3)\cup(3,+\infty)\end{array}\right\}$$

$$\lim_{x\to 3} f(x)=\lim_{x\to 3}\left[g(x)\left(\sqrt{x+6}-3\right)+x\right]=\lim_{x\to 3} g(x)\left(\sqrt{x+6}-3\right)+\lim_{x\to 3} x=$$

$$\lim_{x\to 3} g(x)\lim_{x\to 3}\left(\sqrt{x+6}-3\right)+3=1\bullet 0+3=3$$

$$(\mathrm{II})\ \frac{f(x)-3}{x^2-9} \overset{f(x)=g(x)\left(\sqrt{x+6}-3\right)+x}{=} \frac{g(x)\left(\sqrt{x+6}-3\right)+x-3}{x^2-9}=$$

$$\frac{\sqrt{x+6}-3}{x^2-9}g(x)+\frac{x-3}{x^2-9}=\frac{\left(\sqrt{x+6}-3\right)\left(\sqrt{x+6}+3\right)}{\left(x^2-3^2\right)\left(\sqrt{x+6}+3\right)}g(x)+\frac{x-3}{x^2-3^2}=$$

$$\frac{\left(\sqrt{x+6}\right)^2-3^2}{(x-3)(x+3)\left(\sqrt{x+6}+3\right)}g(x)+\frac{\cancel{x-3}}{\cancel{(x-3)}(x+3)}=$$

$$\frac{\cancel{x-3}}{\cancel{(x-3)}(x+3)\left(\sqrt{x+6}+3\right)}g(x)+\frac{1}{x+3}=\frac{1}{(x+3)\left(\sqrt{x+6}+3\right)}g(x)+\frac{1}{x+3}$$

$$\lim_{x\to 3}\frac{f(x)-3}{x^2-9}=\lim_{x\to 3}\left[\frac{1}{(x+3)\left(\sqrt{x+6}+3\right)}g(x)+\frac{1}{x+3}\right]=$$

$$\lim_{x\to 3}\frac{1}{(x+3)\left(\sqrt{x+6}+3\right)}g(x)+\lim_{x\to 3}\frac{1}{x+3}=\lim_{x\to 3}\frac{1}{(x+3)\left(\sqrt{x+6}+3\right)}\lim_{x\to 3} g(x)+\frac{1}{6}$$

$$=\frac{1}{6\left(\sqrt{9}+3\right)}+\frac{1}{6}=\frac{1}{6\bullet 6}+\frac{1}{6}=\frac{1}{36}+\frac{6}{36}=\frac{7}{36}$$

$$(\mathrm{III})\ \frac{f^2(x)-9}{f^3(x)-3f^2(x)+2f(x)-6}=\frac{f^2(x)-3^2}{f^2(x)\left(f(x)-3\right)+2\left(f(x)-3\right)}=$$

$$\frac{\cancel{(f(x)-3)}(f(x)+3)}{\cancel{(f(x)-3)}(f^2(x)+2)}=\frac{f(x)+3}{f^2(x)+2}$$

$$\lim_{x\to 3}\frac{f^2(x)-9}{f^3(x)-3f^2(x)+2f(x)-6}=\lim_{x\to 3}\frac{f(x)+3}{f^2(x)+2}=\frac{\lim_{x\to 3}(f(x)+3)}{\lim_{x\to 3}(f^2(x)+2)}=$$

$$=\frac{\lim_{x\to 3}f(x)+3}{\lim_{x\to 3}f^2(x)+2}=\frac{\lim_{x\to 3}f(x)+3}{\left(\lim_{x\to 3}f(x)\right)^2+2}=\frac{3+3}{3^2+2}=\frac{6}{11}$$

3.

Δίνεται η συνάρτηση $f:[-2,2)\cup(2,+\infty)\to\mathbb{R}$ με $\lim_{x\to 2}\frac{f(x)-x}{\sqrt{x+2}-2}=3$

Να βρεθούν τα όρια :

$(\mathrm{I})\lim_{x\to 2}f(x)\quad (\mathrm{II})\lim_{x\to 2}\frac{f^2(x)-4}{\sqrt{f^2(x)+5}-(x+1)}$

(I) Θεωρώ την συνάρτηση $g(x)=\frac{f(x)-x}{\sqrt{x+2}-2},x\in[-\infty,2)\cup(2,+\infty)$

Τότε θα έχω: $\lim_{x\to 2}g(x)=3$

$$\left\{\begin{matrix} g(x)=\frac{f(x)-x}{\sqrt{x+2}-2} \\ x\in[-2,2)\cup(2,+\infty)\end{matrix}\right\}\Leftrightarrow\left\{\begin{matrix} f(x)-x=g(x)\left(\sqrt{x+2}-2\right) \\ x\in[-2,2)\cup(2,+\infty)\end{matrix}\right\}\Leftrightarrow$$

$$\left\{\begin{matrix} f(x)=g(x)\left(\sqrt{x+2}-2\right)+x \\ x\in[-2,2)\cup(2,+\infty)\end{matrix}\right\}$$

$$\lim_{x\to 2}f(x)=\lim_{x\to 2}\left[g(x)\left(\sqrt{x+2}-2\right)+x\right]=\lim_{x\to 2}g(x)\left(\sqrt{x+2}-2\right)+\lim_{x\to 2}x=$$

$$=\lim_{x\to 2}g(x)\lim_{x\to 2}\left(\sqrt{x+2}-2\right)+4=3\bullet 0+2=2$$

$$(\mathrm{II})\frac{f^2(x)-4}{\sqrt{f^2(x)+5}-(x+1)}=\frac{\left(f^2(x)-2^2\right)\left[\sqrt{f^2(x)+5}+(x+1)\right]}{\left[\sqrt{f^2(x)+5}-(x+1)\right]\left[\sqrt{f^2(x)+5}+(x+1)\right]}=$$

$$\frac{\left(f(x)-2\right)\left(f(x)+2\right)\left(\sqrt{f^2(x)+5}+x+1\right)}{\left(\sqrt{f^2(x)+5}\right)^2-(x+1)^2}=\frac{\left(f(x)-2\right)\left(f(x)+2\right)\left(\sqrt{f^2(x)+5}+x+1\right)}{f^2(x)+5-x^2-2x-1}$$

$$\frac{f(x)-2}{f^2(x)-x^2-2x+4}\left(f(x)+2\right)\left(\sqrt{f^2(x)+5}+x+1\right)$$

$$\frac{f(x)-2}{f^2(x)-x^2-2x+4}\overset{f(x)=g(x)\left(\sqrt{x+2}-2\right)+x}{=}\frac{\left(\sqrt{x+2}-2\right)g(x)+x-2}{\left[\left(\sqrt{x+2}-2\right)g(x)+x\right]^2-x^2-2x+4}=$$

$$=\frac{\left(\sqrt{x+2}-2\right)g(x)+x-2}{\left[\left(\sqrt{x+2}-2\right)g(x)\right]^2+2\left(\sqrt{x+2}-2\right)g(x)x+\cancel{x^2}-\cancel{x^2}-2x+4}=$$

$$=\frac{\frac{\left(\sqrt{x+2}-2\right)\left(\sqrt{x+2}+2\right)}{\sqrt{x+2}+2}g(x)+x-2}{\left[\frac{\left(\sqrt{x+2}-2\right)\left(\sqrt{x+2}+2\right)}{\sqrt{x+2}+2}\right]^2g^2(x)+2\frac{\left(\sqrt{x+2}-2\right)\left(\sqrt{x+2}+2\right)}{\sqrt{x+2}+2}g(x)x-2(x-2)}=$$

$$=\frac{\frac{\left(\sqrt{x+2}\right)^2-4}{\sqrt{x+2}+2}g(x)+x-2}{\left[\frac{\left(\sqrt{x+2}\right)^2-4}{\sqrt{x+2}+2}\right]^2g^2(x)+2\frac{\left(\sqrt{x+2}\right)^2-4}{\sqrt{x+2}+2}g(x)x-2(x-2)}=$$

$$=\frac{\frac{x-2}{\sqrt{x+2}+2}g(x)+x-2}{\left(\frac{x-2}{\sqrt{x+2}+2}\right)^2g^2(x)+2\frac{x-2}{\sqrt{x+2}+2}g(x)x-2(x-2)}=$$

$$=\frac{\cancel{(x-2)}\left(\frac{g(x)}{\sqrt{x+2}+2}+1\right)}{\cancel{(x-2)}\left[\frac{x-2}{\left(\sqrt{x+2}+2\right)^2}g^2(x)+\frac{2xg(x)}{\sqrt{x+2}+2}-2\right]}=$$

$$=\frac{\frac{g(x)}{\sqrt{x+2}+2}+1}{\frac{x-2}{\left(\sqrt{x+2}+2\right)^2}g^2(x)+\frac{2xg(x)}{\sqrt{x+2}+2}-2}$$

$$\lim_{x\to 2}\frac{f(x)-2}{f^2(x)-x^2-2x+4}=\lim_{x\to 2}\frac{\frac{g(x)}{\sqrt{x+2}+2}+1}{\frac{x-2}{\left(\sqrt{x+2}+2\right)^2}g^2(x)+\frac{2xg(x)}{\sqrt{x+2}+2}-2}=$$

$$=\frac{\lim_{x\to 2}\left[\frac{g(x)}{\sqrt{x+2}+2}+1\right]}{\lim_{x\to 2}\left[\frac{x-2}{\left(\sqrt{x+2}+2\right)^2}g^2(x)+\frac{2xg(x)}{\sqrt{x+2}+2}-2\right]}=$$

$$\frac{\lim_{x\to 2}\frac{g(x)}{\sqrt{x+2}+2}+1}{\lim_{x\to 2}\frac{x-2}{\left(\sqrt{x+2}+2\right)^2}g^2(x)+\lim_{x\to 2}\frac{2xg(x)}{\sqrt{x+2}+2}-2}$$

$$=\frac{\frac{2}{\sqrt{4}+2}+1}{\frac{0}{\left(\sqrt{4}+2\right)^2}3^2+\frac{2\bullet 2\bullet 3}{\sqrt{4}+2}-2}=\frac{\frac{1}{2}+1}{3-2}=\frac{3}{2}$$

$$\lim_{x\to 2}\frac{f^2(x)-4}{\sqrt{f^2(x)+5}-(x+1)}=\lim_{x\to 2}\frac{f(x)-2}{f^2(x)-x^2-2x+4}\left(f(x)+2\right)\left(\sqrt{f^2(x)+5}+x+1\right)$$

$$=\lim_{x\to 2}\frac{f(x)-2}{f^2(x)-x^2-2x+4}\lim_{x\to 2}\left(f(x)+2\right)\lim_{x\to 2}\left(\sqrt{f^2(x)+5}+x+1\right)=\frac{3}{2}\bullet 4\bullet 2\bullet 3=24$$

4.

Δίνεται οι συναρτήσεις f, g *με πεδίο ορισμού το* $\mathbb{R}$, *για τις οποίες ισχύει* $\lim_{x\to 2}\left[2f(x)+g(x)\right]=-1$ *και* $\lim_{x\to 2}\left[3f(x)-g(x)\right]=6.$

Ν*α βρεθούν τα όρια*:

$(\mathrm{I})\lim_{x\to 2}f(x),\lim_{x\to 2}g(x)(\mathrm{II})\lim_{x\to 2}\frac{xf(x)-2g(x)-2f(x)+xg(x)}{\sqrt{2x^2+8}-4}$

(I)

$$\begin{cases}\lim\limits_{x\to 2}\left[2f(x)+g(x)\right]=-1\\ \lim\limits_{x\to 2}\left[3f(x)-g(x)\right]=6\end{cases}(+)$$

$$\lim_{x\to 2}\left[2f(x)+g(x)\right]+\lim_{x\to 2}\left[3f(x)-g(x)\right]=-1+6\Rightarrow$$

$$\lim_{x\to 2}\left[2f(x)+g(x)+3f(x)-g(x)\right]=5\Rightarrow\lim_{x\to 2}\left(5f(x)\right)=5\Rightarrow \cancel{5}\lim_{x\to 2}f(x)=\cancel{5}\bullet 1\Rightarrow$$

$$\lim_{x\to 2}f(x)=1$$

$$\begin{cases}\lim\limits_{x\to 2}\left[2f(x)+g(x)\right]=-1\\ \lim\limits_{x\to 2}f(x)=1\end{cases}\Rightarrow\begin{cases}\lim\limits_{x\to 2}\left[2f(x)+g(x)\right]=-1\\ -2\lim\limits_{x\to 2}f(x)=-2\bullet 1\end{cases}\Rightarrow$$

$$\begin{cases}\lim\limits_{x\to 2}\left[2f(x)+g(x)\right]=-1\\ \lim\limits_{x\to 2}\left(-2f(x)\right)=-2\end{cases}(+)$$

$$\lim_{x\to 2}\left[2f(x)+g(x)\right]+\lim_{x\to 2}\left(-2f(x)\right)=-1-2\Rightarrow\lim_{x\to 2}\left[2f(x)+g(x)-2f(x)\right]=-3$$

$$\Rightarrow\lim_{x\to 2}g(x)=-3$$

(II)

$$\frac{xf(x)-2g(x)-2f(x)+xg(x)}{\sqrt{2x^2+8}-4}=\frac{\underbrace{xf(x)-2f(x)}_{\substack{\text{Βγάζω κοινό παράγοντα}\\ \text{το } f(x)}}\ \underbrace{-2g(x)+xg(x)}_{\substack{\text{Βγάζω κοινό παράγοντα}\\ \text{το } g(x)}}}{\sqrt{2x^2+8}-4}=$$

$$\frac{f(x)(x-2)+g(x)(x-2)}{\sqrt{2x^2+8}-4}=\frac{x-2}{\sqrt{2x^2+8}-4}\left(f(x)+g(x)\right)$$

$$\frac{x-2}{\sqrt{2x^2+8}-4}=\frac{(x-2)\left(\sqrt{2x^2+8}+4\right)}{\left(\sqrt{2x^2+8}-4\right)\left(\sqrt{2x^2+8}+4\right)}=\frac{(x-2)\left(\sqrt{2x^2+8}+4\right)}{\left(\sqrt{2x^2+8}\right)^2-16}=$$

$$\frac{(x-2)\left(\sqrt{2x^2+8}+4\right)}{2x^2-8}=\frac{(x-2)\left(\sqrt{2x^2+8}+4\right)}{2\left(x^2-4\right)}=\frac{\cancel{(x-2)}\left(\sqrt{2x^2+8}+4\right)}{2\cancel{(x-2)}(x+2)}=$$

$$=\frac{\sqrt{2x^2+8}+4}{2(x+2)}$$

$$\lim_{x\to 2}\frac{x-2}{\sqrt{2x^2+8}-4}=\lim_{x\to 2}\frac{\sqrt{2x^2+8}+4}{2(x+2)}=\frac{\sqrt{2\bullet 2^2+8}+4}{2(2+2)}=\frac{2\bullet 4}{2\bullet 4}=1$$

$$\lim_{x\to 2}\frac{xf(x)-2g(x)-2f(x)+xg(x)}{\sqrt{2x^2+8}-4}=\lim_{x\to 2}\frac{x-2}{\sqrt{2x^2+8}-4}\left(f(x)+g(x)\right)=$$

$$\lim_{x\to 2}\frac{x-2}{\sqrt{2x^2+8}-4}\lim_{x\to 2}\left(f(x)+g(x)\right)=1\bullet(1-3)=-2$$

ΑΣΚΗΣΕΙΣ

1.

Δίνεται η συνάρτηση $f:[-3,1)\cup(1,+\infty)\to\mathbb{R}$. Να βρεθεί το $\lim_{x\to 1} f(x)$ όταν

$$\lim_{x\to 1}\frac{\left(\sqrt{x+3}-2\right)f(x)}{\sqrt{x^2+8}-3}=2$$

2.

Δίνεται η συνάρτηση $f:[-7,2)\cup(2,+\infty)\to\mathbb{R}$ με $\lim_{x\to 2}\frac{f(x)+x}{\sqrt{x+7}-3}=1$

Να βρεθούν τα όρια:

(I) $\lim_{x\to 2} f(x)$ (II) $\lim_{x\to 2}\frac{f(x)+2}{x^2-4}$ (III) $\lim_{x\to 2}\frac{f^2(x)-4}{f^3(x)+2f^2(x)+3f(x)+6}$

3.

Δίνεται η συνάρτηση $f:[-1,3)\cup(3,+\infty)\to\mathbb{R}$ με $\lim_{x\to 3}\frac{f(x)+x}{\sqrt{x+1}-2}=2$

Να βρεθούν τα όρια: (I) $\lim_{x\to 3} f(x)$ (II) $\lim_{x\to 3}\frac{f^2(x)-9}{\sqrt{f^2(x)+7}-x-1}$

4.

Δίνεται οι συναρτήσεις f,g με πεδίο ορισμού το $\mathbb{R}$, για τις οποίες ισχύει $\lim_{x\to -1}\left[3f(x)+g(x)\right]=2$ και $\lim_{x\to -1}\left[4f(x)-g(x)\right]=5$.

Να βρεθούν τα όρια:

(I) $\lim_{x\to -1} f(x), \lim_{x\to 2} g(x)$

(II) $\lim_{x\to -1}\frac{x^3f(x)+g(x)+f(x)+x^3g(x)}{\sqrt{x^2+8}-3}$